Appendix A

Process states summary

In this dissertation, we emphasize the use of *process states* to describe the internal states of the evolving systems we are interested in. This view is an extension of Miller's *processes-as-formula* interpretation [Mil93, DCS12]. Of course, a process state is not a formula; Section 4.7.2 discusses our emphasis on SLS process states instead of SLS formulas as the fundamental representation of the internal states of evolving systems.

A process state as defined in Chapter 4 has the form $(\Psi; \Delta)_\sigma$, though outside of Chapter 4 we never use the associated substitution σ, writing $(\Psi; \Delta)$ to indicate the empty substitution $\sigma = \cdot$. The first-order context Ψ, which is sometimes omitted, is also called the *LF context* in SLS because Spine Form LF is the first-order term language of SLS (Section 4.1). Δ is a *substructural context*.

A.1 Substructural contexts

A substructural context (written as Δ and occasionally as Ξ) is a sequence of variable bindings $x{:}T\ lvl$ – all the variables x bound in a context must be distinct. In SLS, lvl is either ord (ordered resources), eph (mobile resources, also called ephemeral or linear resources), or $pers$ (persistent resources).

In stable process states, T is usually a *suspended positive atomic proposition* $\langle Q \rangle$. The permeability of a positive atomic proposition (ordered, mobile/linear/ephemeral, or persistent) is one of its intrinsic properties (Section 2.5.4, Section 3.3.2), so we can write $x{:}\langle Q \rangle$ instead of $x{:}\langle Q \rangle\ ord$, $x{:}\langle Q \rangle\ eph$, or $x{:}\langle Q \rangle\ pers$ if the permeability of Q is known from the context. So the encoding of the string [< > ([ ]), described in the introduction as

$$\mathsf{left(sq)}\ \ \mathsf{left(an)}\ \ \mathsf{right(an)}\ \ \mathsf{left(pa)}\ \ \mathsf{left(sq)}\ \ \mathsf{right(sq)}\ \ \mathsf{right(pa)}$$

is more properly described as

$$x_1{:}\langle \mathsf{left}\ \mathsf{sq} \rangle,\ \ x_2{:}\langle \mathsf{left}\ \mathsf{an} \rangle,\ \ x_3{:}\langle \mathsf{right}\ \mathsf{an} \rangle,\ \ x_4{:}\langle \mathsf{left}\ \mathsf{pa} \rangle,\ \ x_5{:}\langle \mathsf{left}\ \mathsf{sq} \rangle,\ \ x_6{:}\langle \mathsf{right}\ \mathsf{sq} \rangle,\ \ x_7{:}\langle \mathsf{right}\ \mathsf{pa} \rangle$$

We write $x_1{:}\langle \mathsf{left}\ \mathsf{sq} \rangle$ instead of $x_1{:}\langle \mathsf{left}\ \mathsf{sq} \rangle\ ord$ above, leaving implicit the fact that left and right are ordered predicates.

It is also possible, in *nested* SLS specifications (Section 5.1, Section 6.1), to have variable bindings $x{:}A^- \, \mathit{ord}$, $x{:}A^- \, \mathit{eph}$, and $x{:}A^- \, \mathit{pers}$. These nested specifications act much like rules in the SLS signature, though mobile rules ($x{:}A^- \, \mathit{eph}$) can only be used one time, and ordered rules ($x{:}A^- \, \mathit{ord}$) can only be used one time and only in one particular part of the context (Figure 5.2).

Chapter 3 treats substructural contexts strictly as sequences, but in later chapters we treat substructural contexts in a more relaxed fashion, allowing mobile/linear/ephemeral and persistent variable bindings to be tacitly reordered relative to one another other and relative to ordered propositions. In this relaxed reading, $(x_1{:}\langle Q_1\rangle \, \mathit{ord}, x_2{:}\langle Q_2\rangle \, \mathit{ord})$ and $(x_2{:}\langle Q_2\rangle \, \mathit{ord}, x_1{:}\langle Q_1\rangle \, \mathit{ord})$ are not equivalent contexts but $(x_3{:}\langle Q_3\rangle \, \mathit{pers}, x_2{:}\langle Q_2\rangle \, \mathit{ord})$ and $(x_2{:}\langle Q_2\rangle \, \mathit{ord}, x_3{:}\langle Q_3\rangle \, \mathit{pers})$ are.

A *frame* Θ represents a context with a hole in it. The notation $\Theta\{\Delta\}$ tacks the substructural context Δ into the hole in Θ; the context and the frame must have disjoint variable domains for this to make sense. In Chapter 3, frames are interrupted sequences of variable bindings $x_1{:}T_1 \, \mathit{lvl}, \ldots x_n{:}T_n \, \mathit{lvl}, \Box, x_{n+1}{:}T_{n+1} \, \mathit{lvl}, \ldots x_m{:}T_m \, \mathit{lvl}$, where the box represents the hole. In later chapters, this is relaxed in keeping with the relaxed treatment of contexts modulo reordering of mobile and persistent variable bindings.

A.2 Steps and traces

Under focusing, a SLS proposition can correspond to some number of synthetic transitions (Section 2.4, Section 4.2.6). The declaration $\mathsf{rule} : Q_1 \bullet Q_2 \rightarrowtail \{Q_3 \bullet Q_2\}$[1] in an SLS signature Σ, where Q_1 is ordered, Q_2 is mobile, and Q_3 is persistent, is associated with this synthetic transition:

$$\Theta\{x_1{:}\langle Q_1\rangle \, \mathit{ord},\;\; x_2{:}\langle Q_2\rangle \, \mathit{eph}\} \leadsto_\Sigma \Theta\{y_1{:}\langle Q_3\rangle \, \mathit{pers},\;\; y_2{:}\langle Q_2\rangle \, \mathit{eph}\}$$

The variable bindings x_1 and x_2 no longer appear in $\Theta\{y_1{:}\langle Q_3\rangle \, \mathit{pers}, y_2{:}\langle Q_2\rangle \, \mathit{eph}\}$. The proof terms associated with synthetic transitions are *steps* (Section 4.2.6), and the step corresponding to the synthetic transition above is written as $\{y_1, y_2\} \leftarrow \mathsf{rule}\,(x_1 \bullet x_2)$. As described in Section 4.2.6, we can relate the step to the synthetic transition like this:

$$\{y_1, y_2\} \leftarrow \mathsf{rule}\,(x_1 \bullet x_2) :: \Theta\{x_1{:}\langle Q_1\rangle \, \mathit{ord},\;\; x_2{:}\langle Q_2\rangle \, \mathit{eph}\} \leadsto_\Sigma \Theta\{y_1{:}\langle Q_3\rangle \, \mathit{pers},\;\; y_2{:}\langle Q_2\rangle \, \mathit{eph}\}$$

As described in Section 4.2.7, we can also use a more Hoare-logic inspired notation:

$$\begin{array}{l} \quad\;\; \Theta\{x_1{:}\langle Q_1\rangle \, \mathit{ord},\;\; x_2{:}\langle Q_2\rangle \, \mathit{eph}\} \\ \{y_1, y_2\} \leftarrow \mathsf{rule}\,(x_1 \bullet x_2) \\ \quad\;\; \Theta\{y_1{:}\langle Q_3\rangle \, \mathit{pers},\;\; y_2{:}\langle Q_2\rangle \, \mathit{eph}\} \end{array}$$

The reflexive-transitive closure of $\leadsto_\Sigma$ is $\leadsto_\Sigma^*$, and the proof terms witnessing these sequences of synthetic transitions are *traces* $T ::= \diamond \mid S \mid T; T$. *Concurrent equality* (Section 4.3) is an equivalence relation on traces that allows us to rearrange the steps $S_1 = \{P_1\} \leftarrow R_1$ and $S_2 = \{P_2\} \leftarrow R_2$ in a trace when the variables introduced by P_1 (the output interface of S_1, written $S_1{}^\bullet$) are not mentioned in R_2 (the input interface of S_2, written ${}^\bullet S_2$) and vice versa.

[1] This is synonymous with the proposition $Q_1 \bullet Q_2 \rightarrowtail \bigcirc(Q_3 \bullet Q_2)$ (Section 4.2).

Bibliography

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