Chapter 3

Substructural logic

Linear logic is the most famous of the *substructural logics*. Traditional intuitionistic logic, which we call persistent to emphasize the treatment of truth as a persistent and reusable resource, admits the three so-called *structural rules* of weakening (premises need not be used), contraction (premises may be used multiple times) and exchange (the ordering of premises are irrelevant). Substructural logics, then, are logics that do not admit these structural rules – linear logic has only exchange, *affine* logic (which is frequently conflated with linear logic by programming language designers) has exchange and weakening, and *ordered* logic, first investigated as a proof theory by Lambek [Lam58], lacks all three.

Calling logics like linear, affine, and ordered logic sub structural relative to persistent logic is greatly unfair to the substructural logics. Girard's linear logic can express persistent provability using the exponential connective $!A$, and this idea is generally applicable in substructural logics – for instance, it was applied by Polakow and Pfenning to Lambek's ordered logic [PP99]. It is certainly too late to advocate for these logics to be understood as superstructural logics, but that is undoubtedly what they are: generalizations of persistent logic that introduce more expressive power.

In this chapter, we will define a first-order ordered linear logic with a lax connective $\bigcirc A$ in both unfocused (Section 3.1) and focused (Section 3.3) flavors (this logic will henceforth be called OL_3, for ordered linear lax logic). Then, following the structural focalization methodology introduced in the previous chapter, we establish cut admissibility (Section 3.4), and identity expansion (Section 3.5) for focused OL_3; with these results, it is possible to prove the soundness and completeness of focusing (Section 3.6) for OL_3. A fragment of this system will form the basis of the logical framework in Chapter 4, and that framework will, in turn, underpin the rest of this dissertation.

Why present the rich logic OL_3 here if only the fragment detailed in Chapter 4 is needed? There are two main reasons. First, while we will use only a fragment of this logic in Chapter 4, other fragments of the logic may well be interesting and useful for other purposes. Second, the presentation in this chapter, and in particular the discussion of substructural contexts in Section 3.2, introduces a presentation style and infrastructure that I believe will generalize to focused presentations of richer logics, such as the logic of bunched implications [Pym02], non-associative ordered logic (or "rigid logic") [Sim09], subexponential logics [NM09], and so on.

Furthermore, the choice to present a full account of focusing in OL_3 is in keeping with as Andreoli's insistence that we should avoid ambiguity as to whether we are "defining a foundational paradigm or a [logic] programming language (two objectives that should clearly be kept separate)" [And01]. Both the full logic OL_3 and the general methodology followed in this chapter are general, foundational paradigms within which it is possible to instantiate families of logic programming languages and logical frameworks, even though we will focus on a particular logical framework starting in Chapter 4.

3.1 Ordered linear lax logic

Ordered linear logic was the subject of Polakow's dissertation [Pol01]. It extends linear logic with a notion of *ordered resources*. The multiplicative conjunction $A \otimes B$ of linear logic, which represents that we have both the resources to make an A and a B, is replaced in ordered logic by an ordered multiplicative conjunction $A \bullet B$, which represents that we have the resources to make an A, and they're to the left of the resources necessary to make a B. Linear implication $A \multimap B$, which represents a resource that, given the resources necessary to construct an A, can construct a B, splits into two propositions in ordered logic. The proposition $A \rightarrowtail B$ represents a resource that, given the resources necessary to construct an A *to the left*, can construct a B; the proposition $A \twoheadrightarrow B$ demands those resource to its right.

Ordered propositions were used by Lambek to model language [Lam58]. The word "clever" is a adjective that, given a noun to its right, constructs a noun phrase ("ideas" is a noun, and "clever ideas" is a noun phrase). Therefore, the world "clever" can be seen as an ordered resource $\mathsf{Phrase} \twoheadrightarrow \mathsf{NounPhrase}$. Similarly, the word "quietly" is an adverb that, given a verb to its left, constructs a verb phrase ("sleeps" is a verb, and "sleeps quietly" is a verb phrase). Therefore, the word "quietly" can be seen as an ordered resource $\mathsf{Verb} \rightarrowtail \mathsf{VerbPhrase}$. The key innovation made by Polakow and Pfenning was integrating both persistent and linear logic into Lambek's system with the persistent exponential $!A$ and the mobile exponential $¡A$. The latter proposition is pronounced "A mobile" or, whimsically, "gnab A" in reference to the pronunciation of $!A$ as "bang A."

The primary sequent of ordered logic is $\Gamma; \Delta; \Omega \implies A\,\mathit{true}$, which expresses that A is a resource derivable from the persistent resources in Γ, the ephemeral resources in Δ, and the ephemeral, ordered resources in Ω. The persistent context Γ and the linear context Δ are multisets as before (so we think of Δ_1, Δ_2 as being equal to Δ_2, Δ_1, for instance). The ordered context Ω is a sequence of propositions, as in Gentzen's original presentation of sequent calculi, and *not* a multiset. This means that the two ordered contexts Ω_1, Ω_2 and Ω_2, Ω_1 are, in general, not the same.

The presentation of ordered linear lax logic in Figure 3.1 uses an ordered logic adaptation of the matching constructs introduced in Section 2.6; all the left rules in that figure use the construct $\Gamma; \Delta; \Omega_L/A/\Omega_R \Longrightarrow U$, which matches the sequent $\Gamma; \Delta'; \Omega' \Longrightarrow U$

* if $\Omega' = (\Omega_L, A, \Omega_R)$ and $\Delta' = \Delta$;
* if $\Omega' = (\Omega_L, \Omega_R)$ and $\Delta' = (\Delta, A)$;
* or if $\Omega' = (\Omega_L, \Omega_R)$, $\Delta' = \Delta$, and $A \in \Gamma$.

As in the alternative presentation of linear logic where copy was admissible, both the copy rule and a rule Polakow called *place* are admissible in the logic described in Figure 3.1.

$$\frac{\Gamma, A; \Delta; \Omega_L, A, \Omega_R \Longrightarrow U}{\Gamma, A; \Delta; \Omega_L, \Omega_R \Longrightarrow U}\ \mathit{copy} \qquad \frac{\Gamma; \Delta; \Omega_L, A, \Omega_R \Longrightarrow U}{\Gamma; \Delta, A; \Omega_L, \Omega_R \Longrightarrow U}\ \mathit{place}$$

When this notation is used in the rule id, the meaning is that the atomic proposition p is either the only thing in the ordered context alongside an empty linear context, or else it is the only thing in the linear context alongside an empty ordered context, or else both the linear and ordered contexts are empty and p is present in the persistent context. This notion will be called *sole membership* in Section 3.2.1.

Ordered linear lax logic also encompasses Fairtlough and Mendler's lax logic [FM97] as reconstructed by Pfenning and Davies [PD01] and adapted to linear logic is the basis of the CLF logical framework [WCPW02]. The judgment $A\ \mathit{lax}$ is the foundation of ordered logic, and is usually interpreted as truth under some unspecified constraint or as a weaker version of truth: if we know $A\ \mathit{true}$ then we can conclude $A\ \mathit{lax}$:

$$\frac{\Gamma; \Delta; \Omega \Longrightarrow A\ \mathit{true}}{\Gamma; \Delta; \Omega \Longrightarrow A\ \mathit{lax}}\ \mathit{lax}$$

If we know $A\ \mathit{lax}$, on the other hand, we cannot prove $A\ \mathit{true}$, though we can prove $\bigcirc A\ \mathit{true}$, where $\bigcirc A$ is the propositional internalization of the lax judgment (rule $\bigcirc_R$ in Figure 3.1).

Lax truth is handled with the use of matching constructs, thereby making the rule lax rule above admissible just like *copy* and *place* are admissible. We write all the right rules in Figure 3.1 with a construct $\Gamma; \Delta; \Omega \Longrightarrow A\ \mathit{lvl}$ that matches both sequents of the form $\Gamma; \Delta; \Omega \Longrightarrow A\ \mathit{true}$ and sequents of the form $\Gamma; \Delta; \Omega \Longrightarrow A\ \mathit{lax}$ – in other words, we treat lvl as a metavariable ("level") that stands for judgments true or lax on the right. The use of this construct gives us right rules for $A \oplus B$ that look like this:

$$\frac{\Gamma; \Delta; \Omega \Longrightarrow A\ \mathit{true}}{\Gamma; \Delta; \Omega \Longrightarrow A \oplus B\ \mathit{lvl}}\ \oplus_{R1} \qquad \frac{\Gamma; \Delta; \Omega \Longrightarrow B\ \mathit{true}}{\Gamma; \Delta; \Omega \Longrightarrow A \oplus B\ \mathit{lvl}}\ \oplus_{R2}$$

The metavariable U is even more generic, standing for an arbitrary succedent $A\ \mathit{true}$ or $A\ \mathit{lax}$. Putting the pieces of ordered linear lax logic together into one sequent calculus in Figure 3.1 is a relatively straightforward proof-theoretic exercise; the language of propositions is as follows:

$$\begin{array}{lll} \text{Propositions} & A, B, C ::= p \mid {¡}A \mid !A \mid \bigcirc A \mid \\ & \quad \mathbf{1} \mid A \bullet B \mid A \rightarrowtail B \mid A \twoheadrightarrow B \mid \mathbf{0} \mid A \oplus B \mid \top \mid A \mathbin{\&} B \mid \\ & \quad \exists a{:}\tau.A \mid \forall a{:}\tau \mid t \doteq_\tau s \end{array}$$

The connectives and $\mathbf{1}$, $\mathbf{0}$, $A \oplus B$, $\top$, and $A \mathbin{\&} B$ were not mentioned above but are closely analogous to their counterparts in linear logic. The first-order connectives $\exists a{:}\tau.A$, $\forall a{:}\tau$, and $t \doteq_\tau s$ will be discussed presently.

Atomic propositions

$$\frac{}{\Gamma; \cdot; /p/ \Longrightarrow p\; lvl}\; id$$

Exponentials

$$\frac{\Gamma; \Delta; \cdot \Longrightarrow A\; true}{\Gamma; \Delta; \cdot \Longrightarrow {¡}A\; lvl}\; {¡}_R \qquad \frac{\Gamma; \Delta, A; \Omega_L, \Omega_R \Longrightarrow U}{\Gamma; \Delta; \Omega_L/{¡}A/\Omega_R \Longrightarrow U}\; {¡}_L$$

$$\frac{\Gamma; \cdot; \cdot \Longrightarrow A\; true}{\Gamma; \cdot; \cdot \Longrightarrow !A\; lvl}\; !_R \qquad \frac{\Gamma, A; \Delta; \Omega_L, \Omega_R \Longrightarrow U}{\Gamma; \Delta; \Omega_L/!A/\Omega_R \Longrightarrow U}\; !_L$$

$$\frac{\Gamma; \Delta; \Omega \Longrightarrow A\; lax}{\Gamma; \Delta; \Omega \Longrightarrow \bigcirc A\; lvl}\; \bigcirc_R \qquad \frac{\Gamma; \Delta; \Omega_L, A, \Omega_R \Longrightarrow C\; lax}{\Gamma; \Delta; \Omega_L/\bigcirc A/\Omega_R \Longrightarrow C\; lax}\; \bigcirc_L$$

Multiplicative connectives

$$\frac{}{\Gamma; \cdot; \cdot \Longrightarrow \mathbf{1}\; lvl}\; \mathbf{1}_R \qquad \frac{\Gamma; \Delta; \Omega_L, \Omega_R \Longrightarrow U}{\Gamma; \Delta; \Omega_L/\mathbf{1}/\Omega_R \Longrightarrow U}\; \mathbf{1}_L$$

$$\frac{\Gamma; \Delta; \Omega_L \Longrightarrow A\; true \quad \Gamma; \Delta; \Omega_R \Longrightarrow B\; true}{\Gamma; \Delta_1, \Delta_2; \Omega_L, \Omega_R \Longrightarrow A \bullet B\; lvl}\; \bullet_R \qquad \frac{\Gamma; \Delta; \Omega_L, A, B, \Omega_R \Longrightarrow U}{\Gamma; \Delta; \Omega_L/A \bullet B/\Omega_R \Longrightarrow U}\; \bullet_L$$

$$\frac{\Gamma; \Delta; A, \Omega \Longrightarrow B\; true}{\Gamma; \Delta; \Omega \Longrightarrow A \rightarrowtail B\; lvl}\; \rightarrowtail_R \qquad \frac{\Gamma; \Delta_A; \Omega_A \Longrightarrow A\; true \quad \Gamma; \Delta; \Omega_L, B, \Omega_R \Longrightarrow U}{\Gamma; \Delta_A, \Delta; \Omega_L, \Omega_A/A \rightarrowtail B/\Omega_R \Longrightarrow U}\; \rightarrowtail_L$$

$$\frac{\Gamma; \Delta; \Omega, A \Longrightarrow B\; true}{\Gamma; \Delta; \Omega \Longrightarrow A \twoheadrightarrow B\; lvl}\; \twoheadrightarrow_R \qquad \frac{\Gamma; \Delta_A; \Omega_A \Longrightarrow A\; true \quad \Gamma; \Delta; \Omega_L, B, \Omega_R \Longrightarrow U}{\Gamma; \Delta_A, \Delta; \Omega_L/A \twoheadrightarrow B/\Omega_A, \Omega_R \Longrightarrow U}\; \twoheadrightarrow_L$$

Additive connectives

$$\frac{}{\Gamma; \Delta; \Omega_L/\mathbf{0}/\Omega_R \Longrightarrow U}\; \mathbf{0}_L \qquad \frac{\Gamma; \Delta; \Omega \Longrightarrow A\; true}{\Gamma; \Delta; \Omega \Longrightarrow A \oplus B\; lvl}\; \oplus_{R1} \qquad \frac{\Gamma; \Delta; \Omega \Longrightarrow B\; true}{\Gamma; \Delta; \Omega \Longrightarrow A \oplus B\; lvl}\; \oplus_{R2}$$

$$\frac{\Gamma; \Delta; \Omega_L, A, \Omega_R \Longrightarrow U \quad \Gamma; \Delta; \Omega_L, B, \Omega_R \Longrightarrow U}{\Gamma; \Delta; \Omega_L/A \oplus B/\Omega_R \Longrightarrow U}\; \oplus_L$$

$$\frac{}{\Gamma; \Delta; \Omega \Longrightarrow \top\; lvl}\; \top_R \qquad \frac{\Gamma; \Delta; \Omega \Longrightarrow A\; true \quad \Gamma; \Delta; \Omega \Longrightarrow B\; true}{\Gamma; \Delta; \Omega \Longrightarrow A \,\&\, B\; lvl}\; \&_R$$

$$\frac{\Gamma; \Delta; \Omega_L, A, \Omega_R \Longrightarrow U}{\Gamma; \Delta; \Omega_L/A \,\&\, B/\Omega_R \Longrightarrow U}\; \&_{L1} \qquad \frac{\Gamma; \Delta; \Omega_L, B, \Omega_R \Longrightarrow U}{\Gamma; \Delta; \Omega_L/A \,\&\, B/\Omega_R \Longrightarrow U}\; \&_{L2}$$

Figure 3.1: Propositional ordered linear lax logic

$$\frac{\Psi \vdash t : \tau \quad \Gamma; \Delta; \Omega \Longrightarrow [t/a]B \; \mathit{true}}{\Psi; \Gamma; \Delta; \Omega \Longrightarrow \exists a{:}\tau.B \; \mathit{lvl}} \exists_R \qquad \frac{\Psi, a{:}\tau; \Gamma; \Delta; \Omega_L, B, \Omega_R \Longrightarrow U}{\Psi; \Gamma; \Delta; \Omega_L / \exists a{:}\tau.B / \Omega_R \Longrightarrow U} \exists_L$$

$$\frac{\Psi, a{:}\tau; \Gamma; \Delta; \Omega \Longrightarrow B \; \mathit{true}}{\Psi; \Gamma; \Delta; \Omega \Longrightarrow \forall a{:}\tau.B \; \mathit{lvl}} \forall_R \qquad \frac{\Psi \vdash t : \tau \quad \Psi; \Gamma; \Delta; \Omega_L, [t/a]B, \Omega_R \Longrightarrow U}{\Psi; \Gamma; \Delta; \Omega_L / \forall a{:}\tau.B / \Omega_R \Longrightarrow U} \forall_L$$

$$\frac{}{\Psi; \Gamma; \Delta; \Omega \Longrightarrow t \doteq_\tau t \; \mathit{lvl}} \doteq_R$$

$$\frac{\forall (\Psi' \vdash \sigma : \Psi). \quad \sigma t = \sigma s \quad \longrightarrow \quad \Psi'; \sigma\Gamma; \sigma\Delta; \sigma\Omega_L, \sigma\Omega_R \Longrightarrow \sigma U}{\Psi; \Gamma; \Delta; \Omega_L / t \doteq_\tau s / \Omega_R \Longrightarrow U} \doteq_L$$

Figure 3.2: First-order ordered linear lax logic

3.1.1 First-order logic

The presentation in Figure 3.1 is propositional; by uniformly adding a first-order context Ψ to all sequents it can be treated as first-order. We define quantification (existential and universal), as well as first-order equality,[1] in Figure 3.2.

The equality proposition $t \doteq_\tau s$ is an interesting addition to our presentation of the logic, and will be present in the framework SLS defined in the next chapter, albeit in a highly restricted form. Equality in SLS will be used primarily in the logical transformations presented in Chapter 7 and in the program analysis methodology in Chapter 8. The left rule for equality $t \doteq_\tau s$ has a higher-order premise, in the sense that it reflects over the definition of simultaneous term substitutions $\Psi' \vdash \sigma : \Psi$ and over the syntactic equality judgment for first-order terms $t = s$. We used this exact style of presentation previously in [SP11], but the approach is based on Schroeder-Heister's treatment of definitional reflection [SH93].

In one sense, the left rule $\doteq_L$ is actually a rule schema: there is one premise for each substitution σ that is a unifier for t and s (a unifier is any substitution σ that makes σt and σs syntactically identical). When we induct over the structure of proofs, there is correspondingly one smaller subderivation for each unifying substitution. By this reading, $\doteq_L$ is a rule that, in general, will have countably many premises; in the case of a trivially satisfiable equality problem like $x \doteq_\tau x$ it will have one premise for each well-formed substitution that substitutes a term of the appropriate type for x. However, as suggested by Zeilberger [Zei08], it is more auspicious to take the higher-order formulation at face value: the premise is actually a (meta-level) mapping – a function – that takes a substitution σ, the codomain Ψ' of that substitution, and any evidence necessary to show that σ unifies t and s and returns a derivation of $\Psi'; \sigma\Gamma; \sigma\Delta; \sigma\Omega_L, \sigma\Omega_R \Longrightarrow \sigma U$. When we induct over the structure of proofs, the result of applying any unifying substitution to this function is a smaller subderivation for the purpose of invoking the induction hypothesis. This functional interpretation will be reflected in the proof terms we assign to focused OL_3 in Section 3.3.3.

There are two important special cases. First, an unsatisfiable equation on the left implies a contradiction, and makes the left rule for equality equivalent (at the level of provability) to one

[1] That is, equality of terms from the domain of first-order quantification.

with no premises. For instance, this means that

$$\frac{\textit{no unifier for } t \textit{ and } s}{\Psi; \Gamma; \Delta; \Omega_L / t \doteq_\tau s / \Omega_R \Longrightarrow U} \doteq_{no}$$

is derivable – a *unifier* is just a substitution σ such that σt and σs are syntactically identical. The other important special case is when t and s have a *most general unifier* σ_{mgu}, which just means that for all $\Psi' \vdash \sigma : \Psi$ such that $\sigma t = \sigma s$, it is the case that $\sigma = \sigma' \circ \sigma_{mgu}$ for some σ'.[2] In this case, the left rule for equality is equivalent (again, at the level of determining which sequents are provable) to the following rule:

$$\frac{\sigma = mgu(t, s) \quad \Psi' \vdash \sigma : \Psi \quad \Psi'; \sigma\Gamma; \sigma\Delta; \sigma\Omega_L, \sigma\Omega_R \Longrightarrow \sigma U}{\Psi; \Gamma; \Delta; \Omega_L / t \doteq_\tau s / \Omega_R \Longrightarrow U} \doteq_{yes}$$

Therefore, given a first-order domain in which any two terms are decidably either non-unifiable or unifiable with a most general unifier, we can choose to define the logic with two rules $\doteq_{no}$ and $\doteq_{yes}$; the resulting logic will be equivalent, at the level of derivable sequents, to the logic defined with the $\doteq_L$ rule.

We have not yet thoroughly specified the type and term structure of first-order individuals; in Section 4.1 we clarify that these types and terms will actually be types and terms of Spine Form LF. This does mean that we will have to pay attention, in the proofs of this chapter, to the fact that the types of first-order terms τ are *dependent types* that may include terms t. Particularly relevant in this chapter will be simultaneous substitutions σ: the judgment $\Psi' \vdash \sigma : \Psi$ expresses that σ can map terms and propositions defined in the context Ψ (the domain of the substitution) to terms and propositions defined in the context Ψ' (the range of the substitution). Simultaneous substitutions are defined more carefully in Section 4.1.2 and in [NPP08].

3.2 Substructural contexts

First-ordered linear lax logic has a lot of contexts – the persistent context Γ, the linear context Δ, the ordered context Ω, and the first-order context Ψ. In most presentations of substructural logics, the many contexts primarily serve to obscure the logic's presentation and ensure that the LaTeX code of figures and displays remains permanently unreadable. And there are yet more contexts we might want to add, such as the affine contexts present in the Celf implementation [SNS08].

In this section, we will consider a more compact way of dealing with the contexts that we interpret as containing resources (persistent, affine, linear, or ordered resources), though we choose to maintain the distinction between resource contexts and first-order variable contexts Ψ. The particular way we define substructural contexts can be generalized substantially: it would be possible to extend this presentation to the affine exponential $@A$, and we conjecture that the subexponentials discussed by Nigam and Miller [NM09] – as well as richer logics like the logic of bunched implications [Pym02] – could be given a straightforward treatment using this notation.

[2] Where $\circ$ is composition – $(\sigma' \circ \sigma_{mgu})t = \sigma'(\sigma_{mgu} t)$.

We write unified substructural contexts as either Δ or Ξ, preferring the latter when there is a chance of confusing them with linear contexts Δ. For the purposes of encoding OL_3, we can see these contexts as sequences of variable declarations, defined by the grammar

$$\Xi ::= \cdot \mid \Xi, x{:}T\ ord \mid \Xi, x{:}T\ eph \mid \Xi, x{:}T\ pers$$

where each of the *variables* x are distinct, so that the context also represents a finite map from variables x to *judgments* $T\ lvl$, where lvl is either ord, eph, or $pers$. By separating out a substructural context into three subsequences of persistent, linear, and ordered judgments, we can recover the presentations of contexts for OL_3 given in Figure 3.1. We will use this observation in an informal way throughout the chapter, writing $\Xi = \Gamma; \Delta; \Omega$.

The domain represented by the metavariable T is arbitrary: when discussing the unfocused logic given in Figure 3.1, T varies over unpolarized propositions A, but when discussing a focused logic in Section 3.3 it will vary over stable negative propositions A^-, positive suspended propositions $\langle A^+ \rangle$, focused negative propositions $[A^-]$, and inverting positive propositions A^+.

The key innovation in this presentation was already present in the unfocused logic shown in Figure 3.1: we need to differentiate *constructions*, which appear in the premises of rules, and *matching constructs*, which appear in the conclusions of rules. The notation $\Gamma; \Delta; \Omega_L / A \bullet B / \Omega_R$ that appears in the conclusion of $\bullet_L$ is a matching construct; as discussed in Section 3.1, there are multiple ways in which a context $\Gamma'; \Delta'; \Omega'$ could match this context, because $A \bullet B$ could come from any of the three contexts. However, $\Gamma; \Delta; \Omega_L, A, B, \Omega_R$ in the premise of $\bullet_L$ is a construction, and is unambiguously equal to only one context $\Gamma'; \Delta'; \Omega'$ – the one where $\Gamma' = \Gamma$, $\Delta' = \Delta$, and $\Omega' = \Omega_L, A, B, \Omega_R$.

3.2.1 Fundamental operations on contexts

The first fundamental idea we consider is *singleton* contexts. We construct a single-element context by writing $x{:}T\ lvl$. The corresponding matching construct on contexts is $x{:}T$. In unfocused OL_3, we say that Ξ matches $x{:}A$ if its decomposition into persistent, linear, and ordered contexts matches $\Gamma; \cdot; /A/$. Specifically,

Definition 3.1 (Sole membership)**.** *Ξ matches $x{:}T$ if*

* *Ξ contains no linear judgments and contains exactly one ordered judgment $x{:}T\ ord$ (corresponding to the situation where $\Xi = \Gamma; \cdot; T$), or*
* *Ξ contains no ordered judgments and contains exactly one linear judgment $x{:}T\ eph$ (corresponding to the situation where $\Xi = \Gamma; T; \cdot$), or*
* *Ξ contains only persistent judgments, including $x{:}T\ pers$ (corresponding to the situation where $\Xi = \Gamma, T; \cdot; \cdot$).*

Sole membership is related to the initial sequents and the matching construct $\Gamma; \cdot; /A/$ for contexts that was used in Figure 3.1. We could rewrite the *id* rule from that figure as follows:

$$\frac{}{x{:}p \Longrightarrow p\ lvl}\ id$$

As with all rules involving matching constructs in the conclusion, it is fair to view the matching construct as an extra premise; thus, the id rule above is the same as the id rule below:

$$\frac{\Xi\ \textit{matches}\ x{:}p}{\Xi \Longrightarrow p\ \textit{lvl}}\ \textit{id}$$

The second basic operation on contexts requires a new concept, *frames* Θ. Intuitively, we can view a frame as a set of persistent, linear, and ordered contexts where the ordered context is missing a particular piece. We can write this missing piece as a box: $\Gamma; \Delta; \Omega_L, \Box, \Omega_R$. Alternatively, we can think of a frame as a one-hole context or Huet-style zipper [Hue97] over the structure of substructural contexts. We will also think of them morally as linear functions $(\lambda\Xi.\, \Xi_L, \Xi, \Xi_R)$ as in [Sim09].

The construction associated with frames, $\Theta\{\Xi\}$, is just a straightforward operation of filling in the hole or β-reducing the linear function; doing this requires that the variables in Θ and Ξ be distinct. If we think of Θ informally as $\Gamma; \Delta; \Omega_L, \Box, \Omega_R$, then this is *almost* like the operation of filling in the hole, as $\Theta\{x{:}A\ \textit{ord}\} = \Gamma; \Delta; \Omega_L, A, \Omega_R$. The main difference is that we can also use the operation to insert linear propositions ($\Theta\{x{:}A\ \textit{eph}\} = \Gamma; \Delta, A; \Omega_L, \Omega_R$) and persistent propositions ($\Theta\{x{:}A\ \textit{pers}\} = \Gamma, A; \Delta; \Omega_L, \Omega_R$).

The construction associated with frames is straightforward, but the matching construct associated with frames is a bit more complicated. Informally, if we treat linear contexts as multisets and say that $\Xi = \Gamma; \Delta, \Delta'; \Omega_L, \Omega', \Omega_R$, then we can say $\Xi = \Theta\{\!\{\Xi'\}\!\}$ in the case that $\Theta = \Gamma; \Delta; \Omega_L, \Box, \Omega_R$ and $\Xi' = \Gamma; \Delta'; \Omega'$. The sub-context Ξ', then, has been *framed off* from Ξ, its frame is Θ. If we only had ordered judgments $T\ \textit{ord}$, then the framing-off matching construct $\Theta\{\!\{\Xi'\}\!\}$ would be essentially the same as the construction form $\Theta\{\Xi'\}$. However, persistent and linear judgments can be reordered in the process of matching, and persistent judgments always end up in both the frame and the framed-off context.

Definition 3.2 (Framing off). *Ξ matches $\Theta\{\!\{\Xi'\}\!\}$ if the union of the variables in Θ and Ξ' is exactly the variables in Ξ and*

- *if $x{:}T\ \textit{pers} \in \Xi$, then the same variable declaration appears in Θ and Ξ';*
- *if $x{:}T\ \textit{eph} \in \Xi$ or $x{:}T\ \textit{ord} \in \Xi$, then the same variable declaration appears in Θ or Ξ' (but not both);*
- *in both Θ and Ξ', the sequence of variable declarations $x{:}T\ \textit{ord}$ is a subsequence of Ξ; and*
- *if $x{:}T\ \textit{ord} \in \Theta$, then either*
  - *for all $y{:}T'\ \textit{ord} \in \Xi'$, the variable declaration for x appeared before the variable declaration for y in Ξ, or*
  - *for all $y{:}T'\ \textit{ord} \in \Xi'$, the variable declaration for x appeared after the variable declaration for y in Ξ.*

We can use the framing-off notation to describe one of the cut principles for ordered linear lax logic as follows:

$$\frac{\Xi \Longrightarrow A\ \textit{true} \qquad \Theta\{x{:}A\ \textit{true}\} \Longrightarrow C\ \textit{true}}{\Theta\{\!\{\Xi\}\!\} \Longrightarrow C\ \textit{true}}\ \textit{cut}$$

Especially for the eventual proof of this cut principle, it is important to consider that the admissible rule above is equivalent to the following admissible rule, which describes the matching as an explicit extra premise:

$$\frac{\Xi \Longrightarrow A\,\mathit{true} \quad \Theta\{x{:}A\,\mathit{true}\} \Longrightarrow C\,\mathit{true} \quad \Xi'\ \mathit{matches}\ \Theta\{\!\{\Xi\}\!\}}{\Xi' \Longrightarrow C\,\mathit{true}}\ \mathit{cut}$$

An important derived matching construct is $\Theta\{\!\{x{:}T\}\!\}$, which matches Ξ if Ξ matches $\Theta\{\!\{\Xi'\}\!\}$ for some Ξ' such that Ξ' matches $x{:}T$. This notation is equivalent to the matching construct $\Gamma; \Delta; \Omega_L/A/\Omega_R \Longrightarrow U$ from Figure 3.1, which is need to describe almost every left rule for OL_3. Here are three rules given with this matching construct:

$$\frac{\Theta\{y{:}A\,\mathit{ord}\} \Longrightarrow U \quad \Theta\{z{:}B\,\mathit{ord}\} \Longrightarrow U}{\Theta\{\!\{x{:}A \oplus B\}\!\} \Longrightarrow U} \qquad \frac{\Theta\{y{:}A\,\mathit{ord}\} \Longrightarrow U}{\Theta\{\!\{x{:}A \mathbin{\&} B\}\!\} \Longrightarrow U} \qquad \frac{\Theta\{y{:}B\,\mathit{ord}\} \Longrightarrow U}{\Theta\{\!\{x{:}A \mathbin{\&} B\}\!\} \Longrightarrow U}$$

To reemphasize, the reason we use the matching construct $\Theta\{\!\{x{:}A\}\!\}$ in the conclusions of rules is the same reason that we used the notation $\Gamma; \Delta; \Omega_L/A/\Omega_R$ in Figures 3.1 and 3.2: it allows us to generically talk about hypotheses associated with the judgments ord, eph, and pers. The following rules are all derivable using the last of the three rules above:

$$\frac{\Theta\{y{:}B\,\mathit{ord}\} \Longrightarrow U}{\Theta\{x{:}A \mathbin{\&} B\,\mathit{ord}\} \Longrightarrow U} \qquad \frac{\Theta\{y{:}B\,\mathit{ord}\} \Longrightarrow U}{\Theta\{x{:}A \mathbin{\&} B\,\mathit{eph}\} \Longrightarrow U} \qquad \frac{\Theta\{x{:}A \mathbin{\&} B\,\mathit{pers}, y{:}B\,\mathit{ord}\} \Longrightarrow U}{\Theta\{x{:}A \mathbin{\&} B\,\mathit{pers}\} \Longrightarrow U}$$

The consistent use of matching constructs like $\Theta\{\!\{\Delta\}\!\}$ in the conclusion of rules is also what gives us the space to informally treat syntactically distinct sequences of variable declarations as equivalent. As an example, we can think of $(x{:}A\,\mathit{eph}, y{:}B\,\mathit{eph})$ and $(y{:}B\,\mathit{eph}, x{:}A\,\mathit{eph})$ as equivalent by virtue of the fact that they satisfy the same set of matching constructs. Obviously, this means that none of the matching constructs presented in the remainder of this section will observe the ordering of ephemeral or persistent variable declarations.

3.2.2 Multiplicative operations

To describe the multiplicative connectives of OL_3, including the critical implication connectives, we need to have multiplicative operations on contexts. As a construction, Ξ_L, Ξ_R is just the syntactic concatenation of two contexts with distinct variable domains, and the unit $\cdot$ is just the empty sequence. The matching constructs are more complicated to define, but the intuition is, again, uncomplicated: if $\Xi = \Gamma; \Delta, \Delta'; \Omega_L, \Omega_R$, where linear contexts are multisets and ordered contexts are sequences, then $\Xi = \Xi_L, \Xi_R$ if $\Xi_L = \Gamma; \Delta; \Omega_L$ and $\Xi_R = \Gamma; \Delta'; \Omega_R$. Note that here we are using the same notation for constructions and matching constructs: Ξ_L, Ξ_R is a matching construct when it appears in the conclusion of a rule, Ξ_L, Ξ_R is a construction when it appears in the premise of a rule.

Definition 3.3 (Conjunction)**.**
Ξ matches $\cdot$ if Ξ contains only persistent judgments.
Ξ matches Ξ_L, Ξ_R if the union of the variables in Ξ_L and Ξ_R is exactly the variables in Ξ and

* *if $x{:}T\ pers \in \Xi$, then the same variable declaration appears in Ξ_L and Ξ_R;*
* *if $x{:}T\ eph \in \Xi$ or $x{:}T\ ord \in \Xi$, then the same variable declaration appears in Ξ_L or Ξ_R (but not both);*
* *in both Ξ_L and Ξ_R, the sequence of variable declarations $x{:}T\ ord$ is a subsequence of Ξ; and*
* *if $x{:}T\ ord \in \Xi_L$ and $y{:}T'\ ord \in \Xi_R$, then the variable declaration for x appeared before the variable declaration for y in Ξ.*

The constructs for context conjunction are put to obvious use in the description of multiplicative conjunction, which is essentially just the propositional internalization of context conjunction:

$$\frac{\Xi_L \Longrightarrow A\ true \quad \Xi_R \Longrightarrow B\ true}{\Xi_L, \Xi_R \Longrightarrow A \bullet B\ lvl} \quad \frac{\Theta\{y{:}A, z{:}B\} \Longrightarrow U}{\Theta\{\!\{x{:}A \bullet B\}\!\} \Longrightarrow U} \quad \frac{}{\cdot \Longrightarrow \mathbf{1}\ lvl} \quad \frac{\Theta\{\cdot\} \Longrightarrow U}{\Theta\{\!\{x{:}\mathbf{1}\}\!\} \Longrightarrow U}$$

$$\frac{x{:}A\ ord, \Xi \Longrightarrow B\ true}{\Xi \Longrightarrow A \rightarrowtail B\ lvl} \quad \frac{\Xi_A \Longrightarrow A\ true \quad \Theta\{y{:}B\ ord\} \Longrightarrow U}{\Theta\{\!\{\Xi_A, x{:}A \rightarrowtail B\}\!\} \Longrightarrow U}$$

$$\frac{\Xi, x{:}A\ ord \Longrightarrow B\ true}{\Xi \Longrightarrow A \twoheadrightarrow B\ lvl} \quad \frac{\Xi_A \Longrightarrow A\ true \quad \Theta\{y{:}B\ ord\} \Longrightarrow U}{\Theta\{\!\{x{:}A \twoheadrightarrow B, \Xi_A\}\!\} \Longrightarrow U}$$

Implication makes deeper use of context conjunction: Ξ matches $\Theta\{\!\{\Xi_A, x{:}A \rightarrowtail B\}\!\}$ exactly when there exist Ξ' and Ξ'' such that Ξ matches $\Theta\{\!\{\Xi'\}\!\}$, Ξ' matches Ξ_A, Ξ'', and $x{:}A \rightarrowtail B$ matches Ξ''.

3.2.3 Exponential operations

The exponentials ! and ¡ do not have a construction form associated with them, unless we view the singleton construction forms $x{:}T\ pers$ and $x{:}T\ eph$ as being associated with these exponentials. The matching construct is quite simple: Ξ matches $\Xi\restriction_{pers}$ if Ξ contains no ephemeral or ordered judgments – in other words, it says that $\Xi = \Gamma; \cdot; \cdot$. This form can then be used to describe the right rule for $!A$ in unfocused OL_3:

$$\frac{\Xi \Longrightarrow A\ true}{\Xi\restriction_{pers} \Longrightarrow\ !A\ lvl}$$

Similarly, Ξ matches $\Xi\restriction_{eph}$ if Ξ contains no ordered judgments (that is, if $\Xi = \Gamma; \Delta; \cdot$). Ξ always matches $\Xi\restriction_{ord}$; we don't ever explicitly use this construct, but it allows us to generally refer to $\Xi\restriction_{lvl}$ in the statement of theorems like cut admissibility.

The exponential matching constructs don't actually modify contexts in the way other matching constructs do, but this is a consequence of the particular choice of logic we're considering. Given affine resources, for instance, the matching construct associated with the affine connective $@A$ would clear the context of affine facts: Ξ matches $\Xi'\restriction_{pers}$ if Ξ has only persistent and affine resources and Ξ' contains the same persistent resources as Ξ but none of the affine ones.

We can describe a mirror-image operation on succedents U. U matches $U\lfloor^{lax}$ only if it has the form $T\ lax$, and U always matches $U\lfloor^{true}$. The latter matching construct is another degenerate form that similarly allows us to refer to $U\lfloor^{lvl}$ as a generic matching construct. We write $\Delta\restriction_{lvl}$ as

	In the context Δ	As the succedent U
stable propositions	$x{:}A^-\ \mathit{ord}, \mathit{eph}, \mathit{pers}$	$A^+\ \mathit{true}, \mathit{lax}$
suspended propositions (also stable)	$x{:}\langle A^+\rangle\ \mathit{ord}, \mathit{eph}, \mathit{pers}$	$\langle A^-\rangle\ \mathit{true}, \mathit{lax}$
focused propositions	$x{:}[A^-]\ \mathit{ord}$	$[A^+]\ \mathit{true}$
inverting propositions	$x{:}A^+\ \mathit{ord}$	$A^-\ \mathit{true}$

Figure 3.3: Summary of where propositions and judgments appear in OL_3 sequents

a judgment to mean that Δ matches $\Delta\restriction_{\mathit{lvl}}$, and write $U\lfloor^{\mathit{lvl}}$ as a judgment to mean that U matches $U\lfloor^{\mathit{lvl}}$.

The context constructions and context matching constructs that we have given are summarized as follows:

Constructions	$\Delta, \Xi ::= x{:}T\ \mathit{lvl} \mid \Theta\{\Delta\} \mid \cdot \mid \Delta, \Xi$
Matching constructs	$\Delta, \Xi ::= x{:}T \mid \Theta\{\!\{\Delta\}\!\} \mid \cdot \mid \Delta, \Xi \mid \Delta\restriction_{\mathit{lvl}}$

3.3 Focused sequent calculus

A sequent in the focused sequent calculus presentation of OL_3 has the form $\Psi; \Delta \vdash U$, where Ψ is the first-order variable context, Δ is a substructural context as described in the previous section, and U is a succedent. The domain T of the substructural context consists of stable negative propositions A^-, positive suspended propositions $\langle A^+\rangle$, focused negative propositions $[A^-]$, and inverting positive propositions A^+.

The form of the succedent U is $T\ \mathit{lvl}$, where lvl is either true or lax; in this way, U is just a like a special substructural context with exactly one element – we don't need to care about the name of the variable, because there's only one. The domain of T for succedents is complementary to the domain of T for contexts: stable positive propositions A^+, negative suspended propositions $\langle A^-\rangle$, focused positive propositions $[A^+]$, and inverting negative propositions A^-.

Figure 3.3 summarizes the composition of contexts and succedents, taking into account the restrictions discussed below.

3.3.1 Restrictions on the form of sequents

A sequent $\Psi; \Delta \vdash U$ is *stable* when the context Δ and succedent U contain only stable propositions (A^- in the context, A^+ in the succedent) and suspended propositions ($\langle A^+\rangle$ in the context, $\langle A^-\rangle$ in the succedent). We adopt the focusing constraint discussed in Chapter 2: there is only ever at most one focused proposition in a sequent, and if there is focused proposition in the sequent, then the sequent is otherwise stable. A restriction on the rules focus_L and focus_R (presented below in Figure 3.5) is sufficient to enforce this restriction: reading rules from top down, we can only use a rule focus_L or focus_R to prove a stable sequent, and reading rules from bottom up, we can only apply focus_L or focus_R when we are searching for a proof of a stable sequent.

Because there is always a distinct focused proposition in a sequent, we do not need a variable name to reference the focused proposition in a context Δ any more than we need a variable name to reference the unique member of the context-like succedent U. Therefore, we can write $[B^-]\ ord$ instead of $x{:}[B^-]\ ord$. Furthermore, for presentation of focusing that we want to give it suffices to restrict focused propositions and inverting propositions so that they are always associated with the judgment ord (on the left) or $true$ (on the right). With this restriction, we can write $[A^-]$ and $x{:}A^+$ instead of $[A^-]\ ord$ and $x{:}A^+\ ord$ in Δ, and we can write $[A^+]$ and A^- instead of $[A^+]\ true$ and $A^-\ true$ for U.

In a confluent presentation of focused logic like the one given for linear logic in Chapter 2, that would be as far as we could take our simplifications. However, this presentation will use a fixed presentation of logic as described in Section 2.3.8. If there is more than one invertible proposition in a sequent, *only* the leftmost one will be eligible to have a rule or matching applied to it. All the propositions in Δ are treated as being to the left of the succedent U, so we always prioritize inversion on positive propositions in Δ. With this additional restriction, it is always unambiguous which proposition we are referring to in an invertible rule, and we write A^+ instead of $x{:}A^+$ or $x{:}A^+\ ord$.

We will maintain the notational convention (only) within this chapter that first-order variables are written as a, variables associated with stable negative propositions are written as x, and variables associated with suspended positive propositions are written as z.

In summary, the four forms of sequent in focused OL_3, which we define the rules for in Section 3.3.3 below, are:

* Right focused sequents $\Psi; \Delta \vdash [A^+]$ (where Δ is stable, containing only variable declarations $x{:}A^-\ \mathit{lvl}$ or $z{:}\langle A^+\rangle\ \mathit{lvl}$),
* Inversion sequents $\Psi; \Delta \vdash U$ (where Δ contains variable declarations $x{:}A^-\ \mathit{lvl}$, $z{:}\langle A^+\rangle\ \mathit{lvl}$ and inverting positive propositions A^+ and where U is either $A^+\ \mathit{lvl}$, $\langle A^-\rangle\ \mathit{lvl}$, or an inverting negative proposition A^-),
* Stable sequents, the special case of inversion sequents that contain no inverting positive propositions in Δ or inverting negative propositions in U.
* Left focused sequents $\Psi; \Theta\{[A^+]\} \vdash U$ (where Θ and U are stable – Θ contains only variable declarations $x{:}A^-\ \mathit{lvl}$ or $z{:}\langle A^+\rangle\ \mathit{lvl}$ and U is either $A^+\ \mathit{lvl}$ or $\langle A^-\rangle\ \mathit{lvl}$).

3.3.2 Polarized propositions

The propositions of ordered logic are fundamentally sorted into positive propositions A^+ and negative propositions A^-; both classes, and the inclusions between them, are shown in Figure 3.4. The positive propositions have a refinement, *permeable* propositions A^+_{pers}, that is analogous to the refinement discussed for linear logic in Section 2.5.4. There is also a more generous refinement, the *mobile* propositions, A^+_{eph}, for positive propositions that do not mention $\downarrow$ but that may mention ¡. We introduce atomic propositions p^+ that stand for arbitrary positive propositions, mobile atomic propositions p^+_{eph} that stand for arbitrary mobile propositions, and persistent p^+_{pers} that stand for arbitrary permeable propositions. We treat A^+_{ord} and p^+_{ord} as synonymous with A^+ and p^+, respectively, which allows us to generically refer to A^+_{lvl} and p^+_{lvl} in rules like η^+ and in

$$
\begin{array}{rcl}
A^+ & ::= & p^+ \mid p^+_{eph} \mid p^+_{pers} \mid {\downarrow}A^- \mid {¡}A^- \mid {!}A^- \mid \mathbf{1} \mid A^+ \bullet B^+ \mid \mathbf{0} \mid A^+ \oplus B^+ \mid \exists a{:}\tau.A^+ \mid t \doteq_\tau s \\
A^+_{eph} & ::= & p^+_{eph} \mid p^+_{pers} \mid {¡}A^- \mid {!}A^- \mid \mathbf{1} \mid A^+_{eph} \bullet B^+_{eph} \mid \mathbf{0} \mid A^+_{eph} \oplus B^+_{eph} \mid \exists a{:}\tau.A^+_{eph} \mid t \doteq_\tau s \\
A^+_{pers} & ::= & p^+_{pers} \mid {!}A^- \mid \mathbf{1} \mid A^+_{pers} \bullet B^+_{pers} \mid \mathbf{0} \mid A^+_{pers} \oplus B^+_{pers} \mid \exists a{:}\tau.A^+_{pers} \mid t \doteq_\tau s
\end{array}
$$

$$
\begin{array}{rcl}
A^- & ::= & p^- \mid p^-_{lax} \mid {\uparrow}A^+ \mid \bigcirc A^+ \mid A^+ \rightarrowtail B^- \mid A^+ \twoheadrightarrow B^- \mid \top \mid A^- \mathbin{\&} B^- \mid \forall a{:}\tau.A^- \\
A^-_{lax} & ::= & p^-_{lax} \mid \bigcirc A^+ \mid A^+ \rightarrowtail B^-_{lax} \mid A^+ \twoheadrightarrow B^-_{lax} \mid \top \mid A^-_{lax} \mathbin{\&} B^-_{lax} \mid \forall a{:}\tau.A^-_{lax}
\end{array}
$$

Figure 3.4: Propositions of polarized OL_3

the statement of the identity expansion theorem.

Negative propositions also have a refinement, A^-_{lax}, for negative propositions that do not end in an upshift ${\uparrow}A^+$ or in a negative atomic proposition p^-. This is interesting as a formal artifact and there is very little overhead involved in putting it into our development, but the meaning of this syntactic class, as well as the meaning of right-permeable atomic propositions p^-_{lax}, is unclear. Certainly we do *not* want to include such propositions in our logical framework, as to do so would interfere with our development of traces as a syntax for partial proofs in Chapter 4.

The presentation of the exponentials, and the logic that we now present, emphasizes the degree to which the shifts $\uparrow$ and $\downarrow$ have much of the character of exponentials in a focused substructural logic. The upshift ${\uparrow}A^+$ is like an ordered variant of the lax truth $\bigcirc A^+$ that puts no constraints on the form of the succedent, and the downshift ${\downarrow}A^-$ is like an ordered variant of the persistent and linear exponentials ${!}A^-$ and ${¡}A^-$ that puts no constraints on the form of the context. This point is implicit in Laurent's dissertation [Lau02]. In that dissertation, Laurent defines the polarized LLP *without* the shifts $\uparrow$ and $\downarrow$, so that the only connection points between the polarities are the exponentials. Were it not for atomic propositions, the resulting logic would be more persistent than linear, a point we will return to in Section 3.7.

3.3.3 Derivations and proof terms

The multiplicative and exponential fragment of focused OL_3 is given in Figure 3.5, the additive fragment is given in Figure 3.6, and the first-order connectives are treated in Figure 3.7. We follow the convention of using matching constructs in the conclusions of rules and constructions in the premises with the exception of rules that are at the leaves, such as id^+ and $\doteq_R$, where we write out the matching condition as a premise.

These rules are all written with sequents of the form $\Psi; \Delta \vdash E : U$, where E is a *proof term* that corresponds to a derivation of that sequent. Just as sequent forms are divided into the right-focused, inverting, and left-focused sequents, we divide expressions into *values* V, derivations of right-focused sequents; *terms* N, derivations of inverting sequents; and *spines* Sp, derivations

Focus, identity, and atomic propositions

$$\frac{\Delta \vdash V : [A^+]}{\Delta \vdash V : A^+\ \mathit{lvl}}\ \mathit{focus}^*_R \qquad \frac{\Theta\{[A^-]\} \vdash \mathit{Sp} : U}{\Theta\{\!\{x{:}A^-\}\!\} \vdash x \cdot \mathit{Sp} : U}\ \mathit{focus}^*_L$$

$$\frac{\Theta\{z{:}\langle p^+_{\mathit{lvl}}\rangle\ \mathit{lvl}\} \vdash N : U}{\Theta\{\!\{p^+_{\mathit{lvl}}\}\!\} \vdash \langle z\rangle.N : U}\ \eta^+ \qquad \frac{\Delta\ \mathit{matches}\ z{:}\langle A^+\rangle}{\Delta \vdash z : [A^+]}\ \mathit{id}^+$$

$$\frac{\Delta \vdash N : \langle p^-_{\mathit{lvl}}\rangle\ \mathit{lvl}}{\Delta \vdash \langle N\rangle : p^-_{\mathit{lvl}}}\ \eta^- \qquad \frac{\Delta\ \mathit{matches}\ [A^-]}{\Delta \vdash \textsc{nil} : \langle A^-\rangle\ \mathit{lvl}}\ \mathit{id}^-$$

Shifts and modalities

$$\frac{\Delta \vdash N : A^-}{\Delta \vdash {\downarrow}N : [{\downarrow}A^-]}\ {\downarrow}_R \qquad \frac{\Theta\{x{:}A^-\ \mathit{ord}\} \vdash N : U}{\Theta\{\!\{{\downarrow}A^-\}\!\} \vdash {\downarrow}x.N : U}\ {\downarrow}_L$$

$$\frac{\Delta \vdash N : A^-}{\Delta\restriction_{\mathit{eph}} \vdash {¡}N : [{¡}A^-]}\ {¡}_R \qquad \frac{\Theta\{x{:}A^-\ \mathit{eph}\} \vdash N : U}{\Theta\{\!\{{¡}A^-\}\!\} \vdash {¡}x.N : U}\ {¡}_L$$

$$\frac{\Delta \vdash N : A^-}{\Delta\restriction_{\mathit{pers}} \vdash !N : [!A^-]}\ !_R \qquad \frac{\Theta\{x{:}A^-\ \mathit{pers}\} \vdash N : U}{\Theta\{\!\{!A^-\}\!\} \vdash !x.N : U}\ !_L$$

$$\frac{\Delta \vdash N : A^+\ \mathit{true}}{\Delta \vdash {\uparrow}N : {\uparrow}A^+}\ {\uparrow}_R \qquad \frac{\Theta\{A^+\} \vdash N : U}{\Theta\{\!\{[{\uparrow}A^+]\}\!\} \vdash {\uparrow}N : U}\ {\uparrow}_L$$

$$\frac{\Delta \vdash N : A^+\ \mathit{lax}}{\Delta \vdash \{N\} : \bigcirc A^+}\ \bigcirc_R \qquad \frac{\Theta\{A^+\} \vdash N : U}{\Theta\{\!\{[\bigcirc A^+]\}\!\} \vdash \{N\} : U\lfloor^{\mathit{lax}}}\ \bigcirc_L$$

Multiplicative connectives

$$\frac{\Delta\ \mathit{matches}\ \cdot}{\Delta \vdash () : [\mathbf{1}]}\ \mathbf{1}_R \qquad \frac{\Theta\{\cdot\} \vdash N : U}{\Theta\{\!\{\mathbf{1}\}\!\} \vdash ().N : U}\ \mathbf{1}_L$$

$$\frac{\Delta_1 \vdash V_1 : [A^+] \quad \Delta_2 \vdash V_2 : [B^+]}{\Delta_1, \Delta_2 \vdash V_1 \bullet V_2 : [A^+ \bullet B^+]}\ \bullet_R \qquad \frac{\Theta\{A^+, B^+\} \vdash N : U}{\Theta\{\!\{A^+ \bullet B^+\}\!\} \vdash \bullet N : U}\ \bullet_L$$

$$\frac{A^+, \Delta \vdash N : B^-}{\Delta \vdash \lambda^< N : A^+ \rightarrowtail B^-}\ {\rightarrowtail}_R \qquad \frac{\Delta_A \vdash V : [A^+] \quad \Theta\{[B]\} \vdash \mathit{Sp} : U}{\Theta\{\!\{\Delta_A, [A \rightarrowtail B]\}\!\} \vdash V^< \mathit{Sp} : U}\ {\rightarrowtail}_L$$

$$\frac{\Delta, A^+ \vdash N : B^-}{\Delta \vdash \lambda^> N : A^+ \twoheadrightarrow B^-}\ {\twoheadrightarrow}_R \qquad \frac{\Delta_A \vdash V : [A^+] \quad \Theta\{[B]\} \vdash \mathit{Sp} : U}{\Theta\{\!\{[A \twoheadrightarrow B], \Delta_A\}\!\} \vdash V^> \mathit{Sp} : U}\ {\twoheadrightarrow}_L$$

Figure 3.5: Multiplicative, exponential fragment of focused OL_3 (contexts Ψ suppressed)

$$\frac{}{\Theta\{\!\{\mathbf{0}\}\!\} \vdash \text{ABORT} : U}\ \mathbf{0}_L \qquad \frac{\Delta \vdash V : [A^+]}{\Delta \vdash \text{INL}(V) : [A^+ \oplus B^+]}\ \oplus_{R1} \qquad \frac{\Delta \vdash V : [B^+]}{\Delta \vdash \text{INR}(V) : [A^+ \oplus B^+]}\ \oplus_{R2}$$

$$\frac{\Theta\{A^+\} \vdash N_1 : U \quad \Theta\{B^+\} \vdash N_2 : U}{\Theta\{\!\{A^+ \oplus B^+\}\!\} \vdash [N_1, N_2] : U}\ \oplus_L$$

$$\frac{}{\Delta \vdash \top : \top}\ \top_R \qquad \frac{\Delta \vdash N_1 : A^- \quad \Delta \vdash N_2 : B^-}{\Delta \vdash N_1 \mathbin{\&} N_2 : A^- \mathbin{\&} B^-}\ \&_R$$

$$\frac{\Theta\{[A^-]\} \vdash Sp : U}{\Theta\{\![A^- \mathbin{\&} B^-]\!\} \vdash \pi_1; Sp : U}\ \&_{L1} \qquad \frac{\Theta\{[B^-]\} \vdash Sp : U}{\Theta\{\![A^- \mathbin{\&} B^-]\!\} \vdash \pi_2; Sp : U}\ \&_{L2}$$

Figure 3.6: Additive connectives of focused OL_3 (contexts Ψ suppressed)

$$\frac{\Psi \vdash t : \tau \quad \Psi; \Delta \vdash V : [[t/a]A^+]}{\Psi; \Delta \vdash t, V : [\exists a{:}\tau.A^+]}\ \exists_R \qquad \frac{\Psi, a{:}\tau; \Theta\{A^+\} \vdash N : U}{\Psi; \Theta\{\!\{\exists a{:}\tau.A^+\}\!\} \vdash a.N : U}\ \exists_L$$

$$\frac{\Psi, a{:}\tau; \Delta \vdash N : A^-}{\Psi; \Delta \vdash [a].N : \forall a{:}\tau.A^-}\ \forall_R \qquad \frac{\Psi \vdash t : \tau \quad \Psi; \Theta\{[[t/a]A^-]\} \vdash Sp : U}{\Psi; \Theta\{\![\forall a{:}\tau.A^-]\!\} \vdash [t]; Sp : U}\ \forall_L$$

$$\frac{\Delta \ \textit{matches}\ \cdot}{\Psi; \Delta \vdash \text{REFL} : t \doteq_\tau t}\ \doteq_R \qquad \frac{\forall(\Psi' \vdash \sigma : \Psi).\quad \sigma t = \sigma s \quad \longrightarrow \quad \Psi'; \sigma\Theta\{\cdot\} \vdash \phi(\sigma) : \sigma U}{\Psi; \Theta\{\!\{t \doteq_\tau s\}\!\} \vdash \text{UNIF}\,(\mathsf{fn}\,\sigma \Rightarrow \phi(\sigma)) : U}\ \doteq_L$$

Figure 3.7: First-order connectives of focused OL_3

of left-focused sequents. The structure of values, terms, and spines is as follows:

$$\begin{array}{ll}
\text{Values} & V ::= z \mid {\downarrow}N \mid {¡}N \mid {!}N \mid () \mid V_1 \bullet V_2 \mid \text{INL}(V) \mid \text{INR}(V) \mid t, V \mid \text{REFL} \\
\text{Terms} & N ::= V \mid x \cdot Sp \mid \langle z\rangle.N \mid \langle N\rangle \mid {\downarrow}x.N \mid {¡}x.N \mid {!}x.N \mid {\uparrow}N \mid \{N\} \\
 & \quad \mid ().N \mid \bullet N \mid \lambda^{<}N \mid \lambda^{>}N \mid \text{ABORT} \mid [N_1, N_2] \mid \top \mid N_1 \mathbin{\&} N_2 \mid a.N \mid [a].N \\
 & \quad \mid \text{UNIF}\,(\mathsf{fn}\,\sigma \Rightarrow \phi(\sigma)) \\
\text{Spines} & Sp ::= \text{NIL} \mid {\uparrow}N \mid \{N\} \mid V^{<}Sp \mid V^{>}Sp \mid \pi_1; Sp \mid \pi_2; Sp \mid [t]; Sp
\end{array}$$

It is possible to take a "Curry-style" view of expressions as *extrinsically* typed, which means we consider both well-typed and ill-typed expressions; the well-typed expressions are then those for which the sequent $\Psi; \Delta \vdash E : U$ is derivable. However, we will take the "Church-style" view that expressions are intrinsically typed representatives of derivations: that is, $\Psi; \Delta \vdash E : U$ expresses that E is a derivation of the sequent $\Psi; \Delta \vdash U$. To justify this close correspondence, we require the inductive structure of expressions to be faithful to the inductive structure of proofs; this is one reason that we don't introduce the patterns that are common in other proof term assignments for focused logic [WCPW02, LZH08, Kri09]. (In Section 4.2.4, a limited syntax for patterns is introduced as part of the logical framework SLS.)

Proof terms for the left and right identity rules include angle brackets that reflect the notation for suspended propositions: $\langle N\rangle$ for η^- and $\langle z\rangle.N$ for η^+. We distinguish proof terms dealing with existential quantifiers from those dealing with universal quantifiers in a nonstandard way by using square brackets for the latter: $[t]; Sp$ and $[a].N$ represent the left and right rules for universal quantification, whereas $a.N$ and t, V represent the left and right rules for existential quantification. Other than that, the main novelty in the proof term language and in Figures 3.5-3.7 is again the treatment of equality. We represent the proof term corresponding to the left rule for equality as $\textsc{unif}\,(\mathsf{fn}\,\sigma \Rightarrow \phi(\sigma))$, where $(\mathsf{fn}\,\sigma \Rightarrow \phi(\sigma))$ is intended to be a function from unifying substitutions σ to proof terms. This corresponds to the view of the $\doteq_L$ rule that takes the higher-order formulation seriously as a function, and we treat any proof term $\phi(\sigma))$ where σ is a unifying substitution as a subterm of $\textsc{unif}\,(\mathsf{fn}\,\sigma \Rightarrow \phi(\sigma))$.

There are two caveats to the idea that expressions are representatives of derivations. One caveat is that, in order for there to be an actual correspondence between expressions and terms, we need to annotate all variables with the judgment they are associated with, and we need to annotate the proof terms $\textsc{inr}(V)$, $\textsc{inl}(V)$, $\pi_1; Sp$, and $\pi_2; Sp$ with the type of the branch not taken. Pfenning writes these as superscripts [Pfe08], but we will follow Girard in leaving them implicit [GTL89]. The second caveat is that, because we do not explicitly represent the significant bookkeeping associated with matching constructs in proof terms, if $\Psi; \Delta \vdash E : U$, then $\Psi, a{:}\tau; \Delta, x{:}A^+\,\mathit{pers} \vdash E : U$ as well. Therefore, even given appropriate type annotations, when we say that some expression E is a derivation of $\Psi; \Delta \vdash U$, it is only *uniquely* a derivation of that sequent if we account for the implicit bookkeeping on contexts. It is likely that the first caveat can be largely dismissed by treating Figures 3.5-3.7 as bidirectional type system for proof terms. Addressing the second caveat will require a careful analysis of when the bookkeeping on contexts can be reconstructed, which we leave for future work.

The proof terms presented here mirror our formulation of a logical framework in the next chapter. Additionally, working on the level of proof terms allows for a greatly compressed presentation of cut admissibility and identity expansion that emphasizes the computational nature of these proofs: cut admissibility clearly generalizes the *hereditary substitution* operation in so-called *spine form* presentations of LF [CP02], and identity expansion is, computationally, a novel η-expansion property on proof terms. To be fair, much of this compression is due to neglecting the implicit bookkeeping associated with matching constructs, bookkeeping that must be made explicit in proofs like the cut admissibility theorem.

One theorem that takes place entirely at the level of this implicit bookkeeping is the admissible weakening lemma: if Δ' contains only persistent propositions and N is a derivation of $\Psi; \Delta \vdash U$, then N is also a derivation of $\Psi; \Delta, \Delta' \vdash U$. As usual, this proof can be established by straightforward induction on the structure of N.

3.3.4 Variable substitution

The first-order variables introduced by universal quantifiers (on the right) and existential quantifiers (on the left) are proper *variables* in the sense that the meaning of first-order variables is given by substitution [Har12, Chapter 1]. A sequent with free variables is thus a *generic* representative of all the sequents that can be obtained by plugging terms in for those free variables through the operation of substitution. This intuition is formalized by the variable substitution theorem, Theorem 3.4.

Theorem 3.4 (Variable substitution)**.** *If* $\Psi' \vdash \sigma : \Psi$ *and* $\Psi; \Delta \vdash U$*, then* $\Psi'; \sigma\Delta \vdash \sigma U$*.*

Proof. On the level of proof terms, we are given E, a expression corresponding to a derivation of $\Psi; \Delta \vdash U$; we are defining the operation σE, an expression corresponding to a derivation of $\Psi'; \sigma\Delta \vdash \sigma U$.

Propositional fragment For the exponential, multiplicative, and additive fragments, this operation is simple to define at the level of proof terms, and we will omit most of the cases: $\sigma(V_1 \bullet V_2) = \sigma V_1 \bullet \sigma V_2$, $\sigma(\downarrow x.N) = \downarrow x.\sigma N$, and so on. (Note that first-order variables a do not interact with variables x and z in the substructural context.) However, this compact notation does capture a great deal of complexity. In particular, it is important to emphasize that we need lemmas saying that variable substitution is compatible with all the context matching operations from Section 3.2. In full detail, these two simple cases would be:

- $\sigma(V_1 \bullet V_2) = \sigma V_1 \bullet \sigma V_2$

  We are given a proof of $\Psi; \Delta \vdash [A^+ \bullet B^+]$ that ends with the $\bullet_R$ rule; the subderivations are V_1, a derivation of $\Psi; \Delta_1 \vdash [A^+]$, and V_2, a derivation of $\Psi; \Delta_2 \vdash [B^+]$. Furthermore, we know that Δ matches Δ_1, Δ_2. We need a lemma that tells us that $\sigma\Delta$ matches $\sigma\Delta_1, \sigma\Delta_2$; then, by rule $\bullet_R$, it suffices to show that $\Psi'; \sigma\Delta_1 \vdash \sigma A^+$ (which we have by the induction hypothesis on σ and V_1) and that $\Psi'; \sigma\Delta_2 \vdash \sigma B^+$ (which we have by the induction hypothesis on σ and V_2).
- $\sigma(\downarrow x.N) = \downarrow x.\sigma N$

  We are given a proof of $\Psi; \Delta \vdash U$ that ends with $\downarrow_L$; the subderivation is N, a derivation of $\Psi; \Theta\{x{:}A^- \ ord\} \vdash U$. Furthermore, we know that Δ matches $\Theta\{\!\{\downarrow A^-\}\!\}$. We need a lemma that tells us that $\sigma\Delta$ matches $\sigma\Theta\{\!\{\downarrow \sigma A^-\}\!\}$; then, by rule $\downarrow_L$, it suffices to show $\Psi'; \sigma\Theta\{x{:}\sigma A^- \ ord\} \vdash \sigma U$ (which we have by the induction hypothesis on σ and N).

First-order fragment We will present variable substitution on the first-order fragment fully. Note the $\doteq_L$ rule in particular, which does *not* require an invocation of the induction hypothesis. The cases for the $\exists$ quantifier mimic the ones we give for the $\forall$ quantifier, and so the discussion of these cases is omitted.

- $\sigma(t, N) = (\sigma t, \sigma N)$
- $\sigma(a.Sp) = a.(\sigma, a/a)Sp$
- $\sigma(\text{REFL}) = \text{REFL}$
- $\sigma(\text{UNIF}\ (\mathsf{fn}\ \sigma'' \Rightarrow \phi(\sigma''))) = \text{UNIF}\ (\mathsf{fn}\ \sigma' \Rightarrow \phi(\sigma' \circ \sigma))$

  We are given a proof of $\Psi; \Delta \vdash U$ that ends with $\doteq_L$; we know that Δ matches $\Theta\{\!\{t \doteq s\}\!\}$, and the subderivation is ϕ, a function from substitutions $\Psi'' \vdash \sigma'' : \Psi$ that unify t and s to derivations of $\Psi''; \sigma''\Theta\{\cdot\} \vdash \sigma'' U$. We need a lemma that tells us that $\sigma\Delta$ matches $\sigma\Theta\{\!\{\sigma t \doteq \sigma s\}\!\}$; then, by rule $\doteq_L$, it suffices to show that for all $\Psi'' \vdash \sigma' : \Psi'$ that unify σt and σs, there exists a derivation of $\Psi''; \sigma'(\sigma\Theta)\{\cdot\} \vdash \sigma'(\sigma U)$, which is the same thing as a derivation of $\Psi''; (\sigma' \circ \sigma)\Theta\{\cdot\} \vdash (\sigma' \circ \sigma)U$. We have that $\Psi'' \vdash \sigma' \circ \sigma : \Psi$, and certainly $\sigma' \circ \sigma$ unifies t and s, so we can conclude by passing $\sigma' \circ \sigma$ to ϕ.

- $\sigma([a].N) = [a].(\sigma, a/a)N$

  We are given a proof of $\Psi; \Delta \vdash \forall a{:}\tau.A^-$ that ends with $\forall_R$; the subderivation is N, a derivation of $\Psi, a{:}\tau; \Delta \vdash A^-$. Because $\sigma(\forall a{:}\tau.A^-) = \forall a{:}\sigma\tau.(\sigma, a/a)A^-$, by rule $\forall_R$ it suffices to show $\Psi', a{:}\sigma\tau; \sigma\Delta \vdash (\sigma, a/a)A^-$.

  This is the same thing as $\Psi', a{:}\sigma\tau; (\sigma, a/a)\Delta \vdash (\sigma, a/a)A^-$; the result follows by the induction hypothesis on $(\sigma, a/a)$ and N.

- $\sigma([t]; Sp) = [\sigma t]; \sigma Sp$

  We are given a proof of $\Psi; \Delta \vdash U$ that ends with $\forall_L$; the subderivation is Sp, a derivation of $\Psi; \Theta\{[t/a]Sp\} \vdash U$. Furthermore, we know that Δ matches $\Theta\{\!\{[\forall a{:}\tau.A]\}\!\}$. We need a lemma that tells us that $\sigma\Delta$ matches $\sigma\Theta\{\!\{[\forall a{:}\tau.(\sigma, a/a)A^-]\}\!\}$; then, by rule $\forall_L$, it suffices to show $\Psi'; \sigma\Theta\{[[\sigma t/a]((\sigma, a/a)A^-)]\} \vdash \sigma U$.

  This is the same thing as $\Psi'; \sigma\Theta\{[\sigma([t/a]A^-)]\} \vdash \sigma U$; the result follows by the induction hypothesis on σ and Sp.

Note that, in the case for $\forall_R$, the substitution σ was applied to the first-order type τ as well as to the proposition A^-. This alludes to the fact that our first-order terms are dependently typed (Section 4.1). $\square$

Given that we write the constructive content of the variable substitution theorem as σE, where E is an expression, we can also write Theorem 3.4 as an admissible rule in one of two ways, both with and without proof terms:

$$\frac{\Psi' \vdash \sigma : \Psi \quad \Psi; \Delta \vdash E : U}{\Psi'; \sigma\Delta \vdash \sigma E : \sigma U}\ \mathit{varsubst} \qquad \frac{\Psi' \vdash \sigma : \Psi \quad \Psi; \Delta \vdash U}{\Psi'; \sigma\Delta \vdash \sigma U}\ \mathit{varsubst}$$

We will tend towards the expression-annotated presentations, such as the one on the left, in this chapter.

3.3.5 Focal substitution

Both cut admissibility and identity expansion depend on the same focal substitution theorem that was considered for linear logic in Section 2.3.4. Both of these theorems use the compound matching construct $\Theta\{\!\{\Delta{\restriction_{lvl}}\}\!\}$, a pattern that will also be used in the proof of cut admissibility: Δ' matches $\Theta\{\!\{\Delta{\restriction_{lvl}}\}\!\}$ if $\Delta{\restriction_{lvl}}$ (which, again, is a shorthand way of saying Δ matches $\Delta{\restriction_{lvl}}$) and if Δ' matches $\Theta\{\!\{\Delta\}\!\}$.

Theorem 3.5 (Focal substitution)**.**

∗ *If* $\Psi; \Delta \vdash [A^+]$, $\Psi; \Theta\{z{:}\langle A^+\rangle\ lvl\} \vdash U$, *and* Ξ *matches* $\Theta\{\!\{\Delta{\restriction_{lvl}}\}\!\}$, *then* $\Psi; \Xi \vdash U$

∗ *If* $\Psi; \Delta \vdash \langle A^-\rangle\ lvl$, $\Psi; \Theta\{[A^-]\} \vdash U$, Ξ *matches* $\Theta\{\!\{\Delta\}\!\}$, *and* $U{\lfloor^{lvl}}$, *then* $\Psi; \Xi \vdash U$

Proof. The computational content of positive focal substitution is the substitution of a value V for a variable z in an expression E, written $[V/z]E$. As an admissible rule, positive focal substitution is represented as follows:

$$\frac{\Psi; \Delta \vdash V : [A^+] \quad \Psi; \Theta\{z{:}\langle A^+\rangle\, \mathit{lvl}\} \vdash E : U}{\Psi; \Theta\{\!\{\Delta\lceil_{\mathit{lvl}}\}\!\} \vdash [V/z]E : U}\ \mathit{subst}^+$$

The proof of positive focal substitution proceeds by induction over the derivation E containing the suspended proposition. In the case where $E = z$, the derivation z concludes by right focusing on the proposition that we have a focused proof V of, so the result we are looking for is V.

The computational content of negative focal substitution is the substitution of a spine Sp out of an expression E. Sp represents a continuation, and the expression E is waiting on that continuation. As an admissible rule, negative focal substitution is represented as follows:

$$\frac{\Psi; \Delta \vdash E : \langle A^-\rangle\, \mathit{lvl} \quad \Psi; \Theta\{[A^-]\} \vdash Sp : U}{\Psi; \Theta\{\!\{\Delta\}\!\} \vdash [E]Sp : U\lfloor^{\mathit{lvl}}}\ \mathit{subst}^-$$

The proof of negative focal substitution proceeds by induction over the derivation E containing the suspended proposition. In the case where $E = \text{NIL}$, the derivation NIL concludes by left focus on the proposition that we have a spine Sp for, so the result we are looking for is Sp. ☐

Pay attention to the way compound matching constructs are being used. If we separate the substructural context out into its persistent, linear, and ordered constituents, the subst^+ rule can be seen as effectively expressing three admissible principles simultaneously:

* If $\Psi; \Gamma; \Delta; \Omega \vdash [A^+]$ and $\Psi; \Gamma; \Delta'; \Omega_L, \langle A^+\rangle, \Omega_R \vdash U$, then $\Psi; \Gamma; \Delta, \Delta'; \Omega_L, \Omega, \Omega_R \vdash U$.
* If $\Psi; \Gamma; \Delta; \cdot \vdash [A^+_{\mathit{eph}}]$ and $\Psi; \Gamma; \Delta', \langle A^+_{\mathit{eph}}\rangle; \Omega' \vdash U$, then $\Psi; \Gamma; \Delta, \Delta'; \Omega' \vdash U$.
* If $\Psi; \Gamma; \cdot; \cdot \vdash [A^+_{\mathit{pers}}]$ and $\Psi; \Gamma, \langle A^+_{\mathit{pers}}\rangle; \Delta'; \Omega' \vdash U$, then $\Psi; \Gamma; \Delta'; \Omega' \vdash U$.

In negative focal substitution, as in the leftist substitutions of cut admissibility, there is a corresponding use of $U\lfloor^{\mathit{lvl}}$ to capture that we can use a proof of $\langle A^-\rangle\, \mathit{true}$ to discharge a hypothesis of $[A^-]$ in a proof of $C\, \mathit{true}$ or a proof of $C\, \mathit{lax}$, but that a proof of $\langle A^-_{\mathit{lax}}\rangle\, \mathit{lax}$ can only discharge a hypothesis of $[A^-_{\mathit{lax}}]$ in a proof of $C\, \mathit{lax}$.

3.4 Cut admissibility

It is a little wordy to say that, in a context or succedent, the only judgments involving suspensions are $(\langle p^+_{\mathit{pers}}\rangle\, \mathit{pers})$, $(\langle p^+_{\mathit{eph}}\rangle\, \mathit{eph})$, $(\langle p^+\rangle\, \mathit{ord})$, $(\langle p^-\rangle\, \mathit{true})$, and $(\langle p^-_{\mathit{lax}}\rangle\, \mathit{lax})$, but this is a critical precondition of cut admissibility property for focused OL_3. We'll call contexts and succedents with this property *suspension-normal*.

Theorem 3.6 (Cut admissibility)**.** *For suspension-normal* Ψ, A^+, A^-, Δ, Θ, Ξ, *and* U,

1. *If* $\Psi; \Delta \vdash [A^+]$, $\Psi; \Theta\{A^+\} \vdash U$, *and* Ξ *matches* $\Theta\{\!\{\Delta\}\!\}$, *then* $\Psi; \Xi \vdash U$.
2. *If* $\Psi; \Delta \vdash A^-$, $\Psi; \Theta\{[A^-]\} \vdash U$, Δ *is stable, and* Ξ *matches* $\Theta\{\!\{\Delta\}\!\}$, *then* $\Psi; \Xi \vdash U$.

3. *If* $\Psi; \Delta \vdash A^+\ lvl$, $\Psi; \Theta\{A^+\} \vdash U$, Θ *and* U *are stable,*
   Ξ *matches* $\Theta\{\!\{\Delta\}\!\}$, *and* $U\lfloor^{lvl}$, *then* $\Psi; \Xi \vdash U$.
4. *If* $\Psi; \Delta \vdash A^-$, $\Psi; \Theta\{x{:}A^-\ lvl\} \vdash U$, Δ *is stable,*
   and Ξ *matches* $\Theta\{\!\{\Delta\lceil_{lvl}\}\!\}$, *then* $\Psi; \Xi \vdash U$

The four cases of cut admissibility (and their proof below) neatly codify an observation about the structure of cut admissibility proofs made by Pfenning in his work on structural cut elimination [Pfe00]. The first two parts of Theorem 3.6 are the home of the *principal cases* that decompose both derivations simultaneously – part 1 is for positive cut formulas and part 2 is for negative cut formulas. The third part contains all the *left commutative cases* that perform case analysis and induction only on the first given derivation, and the fourth part contains all the *right commutative cases* that perform case analysis and induction only on the second given derivation.

In Pfenning's work on structural cut elimination, this classification of cases was informal, but the structure of our cut admissibility proofs actually isolates the principal, left commutative, and right commutative cases into different parts of the theorem. This separation of cases is the reason why cut admissibility in a focused sequent calculus can use a more refined induction metric than cut admissibility in an unfocused sequent calculus. As noted previously in the proof of Theorem 2.4, the refined induction metric does away with the precondition, essential to Pfenning's proof of structural cut admissibility, that weakening and variable substitution preserve the size of derivations.

Before discussing the proof, it is worth noting that this theorem statement is already a sort of victory. It is an extremely simple statement of cut admissibility for a rather complex logic.

3.4.1 Optimizing the statement of cut admissibility

We will pick the cut admissibility proof from Chaudhuri's dissertation [Cha06] as a representative example of existing work on cut admissibility in focused logics. His statement of cut admissibility for linear logic has 10 parts, which are sorted into five groups. In order to extend his proof structure to handle the extra lax and mobile connectives in OL_3, we would need a dramatically larger number of cases. Furthermore, at a computational level, Chaudhuri's proof requires a lot of code duplication – that is, the proof of two different parts will frequently each require a case that looks essentially the same in both parts.

The structural focalization development in this chapter gives a compact proof of the completeness of focusing that is entirely free of code duplication. A great deal of simplification is due to the use of the matching constructs $\Theta\{\!\{\Delta\lceil_{lvl}\}\!\}$ and $U\lfloor^{lvl}$. Without that notation, part 3 would split into two parts for $true$ and lax and part 4 would split into three parts for ord, eph, and $pers$. The fifth part of the cut admissibility theorem in Section 2.3.6 (Theorem 2.4), which is computationally a near-duplicate of the fourth part of the same theorem, is due to the lack of this device.

Further simplification is due to defining right-focused, inverting, and left-focused sequents as refinements of general sequents $\Psi; \Delta \vdash U$. Without this approach, the statement of part 3 must be split into two parts (for substituting into terms and spines) and the statement of part 4 must be split into three parts (for substituting into values, terms, and spines). Without either of the aforementioned simplifications, we would have 15 parts in the statement of Theorem 3.6 instead of four and twice as many cases that needed to be written down and checked.

Picking a fixed inversion strategy prevents us from having to prove the tedious, quadratically large confluence theorem discussed for linear logic in Section 2.3.8. This confluence theorem is certainly true, and we might want to prove it for any number of reasons, but it is interesting that we can avoid it altogether in our current development. A final improvement in our theorem statement is very subtle: insofar as our goal is to give a short proof of the completeness of focusing that avoids redundancy, the *particular* fixed inversion strategy we choose matters. The proof of Theorem 2.4 duplicates many right commutative cases in both part 1 and part 4 (which map directly onto parts 1 and 4 of Theorem 3.6 above). Our system prioritizes the inversion of positive formulas on the left over the inversion of negative formulas on the right. If we made the opposite choice, as Chaudhuri's system does, then this issue would remain, resulting in code duplication. We get a lot of mileage out of the fact that if $\Xi = \Theta\{A^+\}$ then A^+ unambiguously refers to the left-most proposition in Ξ, and this invariant would no longer be possible to maintain in the proof of cut admissibility if we prioritized inversion of negative propositions on the right.

3.4.2 Proof of cut admissibility, Theorem 3.6

The proof proceeds by lexicographic induction. In parts 1 and 2, the type gets smaller in every call to the induction hypothesis. In part 3, the induction hypothesis is only ever invoked on the same type A^+, and every invocation of the induction hypothesis is either to part 1 (smaller part number) or to part 3 (same part number, first derivation is smaller). Similarly, in part 4, the induction hypothesis is only invoked at the same type A^-, and every invocation of the induction hypothesis is either to part 2 (smaller part number) or to part 4 (same part number, second derivation is smaller).

The remainder of this section will cover each of the four parts of this proof in turn. Most of the theorem will be presented at the level of proof terms, but for representative cases we will discuss what the manipulation of proof terms means in terms of sequents and matching constructs. The computational content of parts 1 and 2 is *principal substitution*, written as $(V \circ N)^{A^+}$ and $(N \circ Sp)^{A^-}$ respectively, the computational content of part 3 is *leftist substitution*, written as $[\![E]\!]^{A^+}N$, and the computational content of part 4 is *rightist substitution*, written as $[\![M/x]\!]^{A^-}E$.

In many cases, we discuss the necessity of constructing certain contexts or frames; in general, we will state the necessary properties of these constructions without detailing the relatively straightforward process of constructing them.

Positive principal substitution

Positive principal substitution encompasses half the *principal cuts* from Pfenning's structural cut admissibility proof – the principal cuts where the principal cut formula is positive. The constructive content of this part is a function $(V \circ N)^{A^+}$ that normalizes a value against a term. Induction is on the structure of the positive type. The admissible rule associated with principal positive substitution is cut^+.

$$\frac{\Psi; \Delta \vdash V : [A^+] \quad \Psi; \Theta\{A^+\} \vdash N : U}{\Psi; \Theta\{\!\{\Delta\}\!\} \vdash (V \circ N)^{A^+} : U}\ cut^+$$

We have to be careful, especially in the positive principal substitution associated with the type $A^+ \bullet B^+$, to maintain the invariant that, in an unstable context, we only ever consider the *leftmost* inverting positive proposition.

In most of these cases, one of the givens is that $\Theta\{A^+\}$ matches $\Theta'\{\!\{A^+\}\!\}$ for some Θ'. Because this implies that $\Theta = \Theta'$, we take the equality for granted rather than mentioning and reasoning explicitly about the premise every time.

– $(z \circ \langle z' \rangle.N_1)^{p^+_{lvl}} = [z/z']N_1$
  We must show $\Psi; \Xi \vdash U$, where
  - Δ matches $z{:}\langle p^+_{lvl} \rangle$,
  - N_1 is a derivation of $\Psi; \Theta\{z'{:}\langle p^+_{lvl} \rangle\ lvl\} \vdash U$,
  - and Ξ matches $\Theta\{\!\{\Delta\}\!\}$.

  Because Δ is suspension-normal, we can derive $\Psi; \Delta \vdash [p^+_{lvl}]$ by id^+, and Ξ matches $\Theta\{\!\{\Delta\lceil_{lvl}\}\!\}$. Therefore, the result follows by focal substitution on z and N_1.

– $(\downarrow M \circ \downarrow x.N_1)^{\downarrow A^-} = [\![M/x]\!]^{A^-} N_1$
– $(¡M \circ ¡x.N_1)^{¡A^-} = [\![M/x]\!]^{A^-} N_1$
– $(!M \circ !x.N_1)^{!A^-} = [\![M/x]\!]^{A^-} N_1$
  We must show $\Psi; \Xi \vdash U$, where
  - Δ matches $\Delta\lceil_{pers}$, M is derivation of $\Psi; \Delta \vdash A^-$,
  - N_1 is a derivation of $\Psi; \Theta\{x{:}A^-\ pers\} \vdash U$,
  - and Ξ matches $\Theta\{\!\{\Delta\}\!\}$.

  Ξ matches $\Theta\{\!\{\Delta\lceil_{pers}\}\!\}$ and Δ is stable (it was in a focused sequent $\Psi; \Delta \vdash\ !M : [!A^-]$), so the result follows by part 4 of cut admissibility on N_1 and M.

– $(() \circ ().N_1)^{\mathbf{1}} = N_1$
– $((V_1 \bullet V_2) \circ (\bullet N_1))^{A^+ \bullet B^+} = (V_2 \circ (V_1 \circ N_1)^{A^+})^{B^+}$
  We must show $\Psi; \Xi \vdash U$, where
  - Δ matches Δ_1, Δ_2,
    V_1 is a derivation of $\Psi; \Delta_1 \vdash [A^+]$, V_2 is a derivation of $\Psi; \Delta_2 \vdash [B^+]$,
  - N_1 is a derivation of $\Psi; \Theta\{A^+, B^+\} \vdash U$,
  - and Ξ matches $\Theta\{\!\{\Delta\}\!\}$.

  We can to construct a frame Θ_B such that $\Theta\{A^+, B^+\} = \Theta_B\{A^+\}$; we're just exchanging the part in the frame with the part not in the frame. We can also construct a second frame, Θ_A, such that 1) Ξ matches $\Theta_A\{\!\{\Delta_2\}\!\}$ and 2) $\Theta_A\{B^+\}$ matches $\Theta_B\{\!\{\Delta_1\}\!\}$.

  Because $\Theta_A\{B^+\}$ matches $\Theta_B\{\!\{\Delta_1\}\!\}$, by the induction hypothesis on V_1 and N_1 we have $(V_1 \circ N_1)^{A^+}$, a derivation of $\Psi; \Theta_A\{B^+\} \vdash U$.

Because Ξ matches $\Theta_A\{\!\{\Delta_2\}\!\}$, by the induction hypothesis on V_2 and $(V_1 \circ N_1)^{A^+}$, we have a derivation of $\Psi; \Xi \vdash U$ as required.

– $(\text{INL}(V_1) \circ [N_1, N_2])^{A^+ \oplus B^+} = (V_1 \circ N_1)^{A^+}$
– $(\text{INR}(V_2) \circ [N_1, N_2])^{A^+ \oplus B^+} = (V_2 \circ N_2)^{B^+}$
– $(t, V_1 \circ a.N_1)^{\exists a:\tau.A^+} = (V_1 \circ [t/a]N_1)^{[t/a]A^+}$

We must show $\Psi; \Xi \vdash U$, where

- $\Psi \vdash t : \tau$, V_1 is a derivation of $\Psi; \Delta \vdash [[t/a]A^+]$,
- N_1 is a derivation of $\Psi, a{:}\tau; \Theta\{A^+\} \vdash U$,
- and Ξ matches $\Theta\{\!\{\Delta\}\!\}$.

By variable substitution on $[t/a]$ and N_1, we have a derivation $[t/a]N_1$ of $\Psi; \Theta\{[t/a]A^+\} \vdash U$. We count $[t/a]A^+$ as being a smaller formula than $\exists a{:}\tau.A^+$, so by the induction hypothesis on V_1 and $[t/a]N_1$, we get a derivation of $\Psi; \Xi \vdash U$ as required.

– $(\text{REFL} \circ \text{UNIF}\,(\mathsf{fn}\,\sigma \Rightarrow \phi(\sigma)))^{t \doteq t} = \phi(\mathsf{id})$

We must show $\Psi; \Xi \vdash U$, where

- Δ matches $\cdot$,
- ϕ is a function from substitutions $\Psi' \vdash \sigma : \Psi$ that unify t and t to derivations of $\Psi; \Theta\{\cdot\} \vdash U$,
- and Ξ matches $\Theta\{\!\{\Delta\}\!\}$.

We simply apply the identity substitution to ϕ to obtain a derivation of $\Psi; \Theta\{\cdot\} \vdash U$. Note that this is not quite the derivation of $\Psi; \Xi \vdash U$ that we need; we need an exchange-like lemma that, given a derivation of $\Psi; \Theta\{\cdot\} \vdash U$ and the fact that Ξ matches $\Theta\{\!\{\cdot\}\!\}$, we can get a proof of $\Psi; \Xi \vdash U$ as we require.

Negative principal substitution

Negative principal substitution encompass all the *principal cuts* from Pfenning's structural cut admissibility proof for which the principal formula is negative. The constructive content of this part is a function $(N \circ Sp)^{A^-}$ that normalizes a term against a spine; a similar function appears as *hereditary reduction* in presentations of hereditary substitution for LF [WCPW02]. Induction is on the structure of the negative type. The admissible rule associated with negative principal substitution is cut^-:

$$\frac{\Psi; \Delta \vdash N : A^- \quad \Psi; \Theta\{[A^-]\} \vdash Sp : U \quad \Delta\ stable_L}{\Psi; \Theta\{\!\{\Delta\}\!\} \vdash (N \circ Sp)^{A^-} : U}\ cut^-$$

– $(\langle N \rangle \circ \text{NIL})^{p^-_{lvl}} = N$

We must show $\Psi; \Xi \vdash U$, where

- N is a derivation of $\Psi; \Delta \vdash \langle p^-_{lvl} \rangle\ lvl$
- $\Theta\{[p^-_{lvl}]\}$ matches $[p^-_{lvl}]$, $U = \langle p^-_{lvl} \rangle\ lvl'$,

- and Ξ matches $\Theta\{\!\{\Delta\}\!\}$.

Because U is suspension-normal, $lvl = lvl'$. A derivation of $\Psi;\Delta \vdash \langle p^-_{lvl}\rangle\, lvl$ is not quite a proof of $\Psi;\Xi \vdash U$, so we need an exchange-like lemma that we can get one from the other.

– $(\uparrow N \circ \uparrow M)^{\uparrow A^+} = [\![N]\!]^{A^+} M$
– $(\{N\} \circ \{M\})^{\bigcirc A^+} = [\![N]\!]^{A^+} M$

We must show $\Psi;\Xi \vdash U$, where

- N is a derivation of $\Psi;\Delta \vdash A^+\ lax$,
- $\Theta\{\bigcirc A^+\}$ matches $\Theta'\{\!\{\bigcirc A^+\}\!\}$, $U\lfloor^{lax}$, Θ' and U are stable, M is a derivation of $\Psi;\Theta'\{A^+\} \vdash U$,
- and Ξ matches $\Theta\{\!\{\Delta\}\!\}$.

Ξ matches $\Theta'\{\!\{\Delta\}\!\}$, so the result follows by part 3 of cut admissibility on N and M.

– $((\lambda^< N) \circ (V^< Sp))^{A^+ \rightarrowtail B^-} = ((V \circ N)^{A^+} \circ Sp)^{B^-}$
– $((\lambda^> N) \circ (V^> Sp))^{A^+ \twoheadrightarrow B^-} = ((V \circ N)^{A^+} \circ Sp)^{B^-}$

We must show $\Psi;\Xi \vdash U$, where

- N is a derivation of $\Psi;\Delta, A^+ \vdash B^-$, Δ is stable (by the fixed inversion invariant – we only invert on the right when there is no further inversion to do on the left),
- $\Theta\{[A^+ \twoheadrightarrow B^-]\}$ matches $\Theta'\{\!\{[A^+ \twoheadrightarrow B^-], \Delta_A\}\!\}$, V is a derivation of $\Psi;\Delta_A \vdash [A^+]$, Sp is a derivation of $\Psi;\Theta'\{[B^-]\} \vdash U$,
- and Ξ matches $\Theta\{\!\{\Delta\}\!\}$.

We can simultaneously view the construction Δ, A^+ as a frame Θ_Δ such that $\Theta_\Delta\{A^+\} = \Delta, A^+$. Note that this is only possible to do because Δ is stable; if there were a non-stable proposition in Δ, the fixed inversion invariant would not permit us to frame off the right-most proposition A^+.

We next construct a context Δ'_A that matches $\Theta_\Delta\{\!\{\Delta_A\}\!\}$ (and also Δ, Δ_A viewed as a matching construct), while simultaneously Ξ matches $\Theta'\{\!\{\Delta'_A\}\!\}$.

By the part 1 of cut admissibility on V and N, we have $(V \circ N)^{A^+}$, a derivation of $\Psi;\Delta'_A \vdash B^-$, and the result then follows by the induction hypothesis on $(V \circ N)^{A^+}$ and Sp.

– $((N_1 \mathbin{\&} N_2) \circ (\pi_1; Sp))^{A^- \& B^-} = (N_1 \circ Sp)^{A^-}$
– $((N_1 \mathbin{\&} N_2) \circ (\pi_2; Sp))^{A^- \& B^-} = (N_2 \circ Sp)^{A^-}$
– $(([a].N) \circ ([t]; Sp))^{\forall a{:}\tau.A^-} = ([t/a]N \circ Sp)^{[t/a]A^-}$

Leftist substitution

In focal substitution, the positive case corresponds to our usual intuitions about substitution and the negative case is strange. In cut admissibility, the situation is reversed: rightist substitutions (considered in Section 3.4.2 below), associated with negative principal cut formals, look like normal substitutions, and the leftist substitutions, considered here, are strange, as they break apart the expression that proves A^+ rather than the term where A^+ appears in the context.

Leftist substitutions encompass all the *left commutative cuts* from Pfenning's structural cut admissibility proof. The constructive content of leftist substitution is a function $[\![E]\!]M$; we say we are *substituting* M *out of* E. Induction is on the first subterm, as we crawl through E looking for places where focus takes place on the right. The admissible rule associated with leftist substitution is $lcut$:

$$\frac{\Psi; \Delta \vdash E : A^+ \, lvl \quad \Psi; \Theta\{A^+\} \vdash M : U \quad \Theta \; stable_L \quad U \; stable_R}{\Psi; \Theta\{\!\{\Delta\}\!\} \vdash [\![E]\!]^{A^+} M : U \lfloor^{lvl}} \; lcut$$

Except for the case where the first given derivation ends in the rule $focus_R$, every case of this theorem involves a left rule. The general pattern for these cases is that Ξ matches $\Theta\{\!\{\Delta\}\!\}$ and Δ matches $\Theta_B\{\!\{x{:}T \; ord\}\!\}$. Θ and Θ_B have the same persistent variables but distinct ephemeral and ordered variables, and we must construct a frame $\Theta{\circ}\Theta_B$ that is effectively the composition of Θ and Θ_B. In cases that we discuss in detail, necessary properties of this composition frame are stated but not proven.

Substitution out of terms

– $[\![V]\!]^{A^+} M = (V \circ M)^{A^+}$

  We must show $\Psi; \Xi \vdash U$, where

  - V is a derivation of $\Psi; \Delta \vdash [A^+]$,
  - M is a derivation of $\Psi; \Theta\{A^+\} \vdash U$,
  - Ξ matches $\Theta\{\!\{\Delta\}\!\}$, and $U\lfloor^{lvl}$

  The result follows from part 1 of cut admissibility on V and M.

– $[\![x \cdot Sp]\!]^{A^+} M = x \cdot ([\![Sp]\!]^{A^+} M)$

  We must show $\Psi; \Xi \vdash U$, where

  - Δ matches $\Theta_B\{\!\{x{:}B^-\}\!\}$, Sp is a derivation of $\Psi; \Theta_B\{[B^-]\} \vdash A^+ \; lvl$,
  - M is a derivation of $\Psi; \Theta\{A^+\} \vdash U$,
  - Ξ matches $\Theta\{\!\{\Delta\}\!\}$, and $U\lfloor^{lvl}$.

  Ξ matches $(\Theta{\circ}\Theta_B)\{\!\{x{:}B^-\}\!\}$ and $(\Theta{\circ}\Theta_B)\{[B^-]\}$ matches $\Theta'\{\!\{\Theta_B\{[B^-]\}\}\!\}$. By the induction hypothesis on Sp and M we have $\Psi; (\Theta{\circ}\Theta_B)\{[B^-]\} \vdash U$, and the required result then follows from rule $focus_L$.

– $[\![\langle z \rangle.N]\!]^{A^+} M = \langle z \rangle.([\![N]\!]^{A^+} M)$

– $[\![{\downarrow}x.N]\!]^{A^+} M = {\downarrow}x.([\![N]\!]^{A^+} M)$

– $[\![¡x.N]\!]^{A^+} M = ¡x.([\![N]\!]^{A^+} M)$

– $[\![!x.N]\!]^{A^+} M = !x.([\![N]\!]^{A^+} M)$

  We must show $\Psi; \Xi \vdash U$, where

  - Δ matches $\Theta_B\{\!\{!B^-\}\!\}$, N is a derivation of $\Psi; \Theta_B\{x{:}B^- \; pers\} \vdash A^+ \; lvl$,
  - M is a derivation of $\Psi; \Theta\{A^+\} \vdash U$,

- Ξ matches $\Theta\{\!\{\Delta\}\!\}$, and $U\lfloor^{lvl}$.

We can construct a Θ' such that $\Theta'\{A^+\} = (\Theta\{A^+\}, x{:}B^-\,pers)$. By admissible weakening, M is a derivation of $\Psi; \Theta'\{A^+\} \vdash U$, too.

Ξ matches $(\Theta{\circ}\Theta_B)\{\!\{!B^-\}\!\}$ and $(\Theta{\circ}\Theta_B)\{x{:}B^-\,pers\}$ matches $\Theta'\{\!\{\Theta_B\{x{:}B^-\,pers\}\}\!\}$. By the induction hypothesis on N and M we have $\Psi; (\Theta{\circ}\Theta_B)\{x{:}B^-\,pers\} \vdash U$, and the required result then follows from rule $!_L$.

- $[\![\bullet N]\!]^{A^+} M = \bullet([\![N]\!]^{A^+} M)$
- $[\![\textsc{abort}]\!]^{A^+} M = \textsc{abort}$
- $[\![[N_1, N_2]]\!]^{A^+} M = [([\![N_1]\!]^{A^+} M), ([\![N_2]\!]^{A^+} M)]$
- $[\![a.N]\!]^{A^+} M = a.([\![N]\!]^{A^+} M)$

We must show $\Psi; \Xi \vdash U$, where

- Δ matches $\Theta_B\{\!\{\exists a{:}\tau.B^+\}\!\}$, N is a derivation of $\Psi, a{:}\tau; \Theta_B\{B^+\} \vdash A^+$,
- M is a derivation of $\Psi; \Theta\{A^+\} \vdash U$,
- Ξ matches $\Theta\{\!\{\Delta\}\!\}$, and $U\lfloor^{lvl}$.

Ξ matches $(\Theta{\circ}\Theta_B)\{\!\{\exists a{:}\tau.B^+\}\!\}$ and $(\Theta{\circ}\Theta_B)\{B^+\}$ matches $\Theta\{\!\{\Theta_B\{B^+\}\}\!\}$. By variable weakening, M is also a derivation of $\Psi, a{:}\tau; \Theta\{A^+\} \vdash U$, so by the induction hypothesis on N and M we have $\Psi, a{:}\tau; (\Theta{\circ}\Theta_B)\{B^+\} \vdash U$, and the required result then follows from rule $\exists_L$.

- $[\![\textsc{unif}\,(\mathsf{fn}\,\sigma \Rightarrow \phi(\sigma))]\!]^{A^+} M = \textsc{unif}\,(\mathsf{fn}\,\sigma \Rightarrow [\![\phi(\sigma)]\!]^{\sigma A^+}(\sigma M))$

We must show $\Psi; \Xi \vdash U$, where

- Δ matches $\Theta_B\{\!\{t \doteq s\}\!\}$, ϕ is a function from substitutions $\Psi' \vdash \sigma : \Psi$ that unify t and s to derivations of $\Psi'; \sigma\Theta_B\{\cdot\} \vdash \sigma A^+$,
- M is a derivation of $\Psi; \Theta\{A^+\} \vdash U$,
- Ξ matches $\Theta\{\!\{\Delta\}\!\}$, and $U\lfloor^{lvl}$.

Ξ matches $(\Theta{\circ}\Theta_B)\{\!\{t \doteq s\}\!\}$, and for any substitution σ, $\sigma U\lfloor^{lvl}$ and $\sigma(\Theta{\circ}\Theta_B)\{\cdot\}$ matches $\sigma\Theta\{\!\{\sigma\Theta_B\{\cdot\}\}\!\}$. By rule $\doteq_L$, it suffices to show that, given an arbitrary substitution $\Psi' \vdash \sigma : \Psi$, there is a derivation of $\Psi'; \sigma(\Theta{\circ}\Theta_B)\{\cdot\} \vdash \sigma U$.

By applying σ to ϕ, we get $\phi(\sigma)$, a derivation of $\Psi'; \sigma\Theta_B\{\cdot\} \vdash \sigma A^+$. We treat σA^+ as having the same size as A^+, and the usual interpretation of higher-order derivations is that $\phi(\sigma)$ is a subderivation of ϕ, so $\phi(\sigma)$ can be used to invoke the induction hypothesis. From variable substitution, we get σM, a derivation of $\Psi'; \sigma\Theta\{\sigma A^+\} \vdash \sigma U$, and then the result follows by the induction hypothesis on $\phi(\sigma)$ and σM.

Substitution out of spines

- $[\![\uparrow N]\!]^{A^+} M = \uparrow([\![N]\!]^{A^+} M)$
- $[\![\{N\}]\!]^{A^+} M = \{[\![N]\!]^{A^+} M\}$

We must show $\Psi; \Xi \vdash U$, where

- Δ matches $\Theta_B\{\!\{\bigcirc B^+\}\!\}, (A^+\ \mathit{lvl})\lfloor^{lax}$,
  N is a derivation of $\Psi; \Theta_B\{A^+\} \vdash U_A$
- M is a derivation of $\Psi; \Theta\{A^+\} \vdash U$,
- Ξ matches $\Theta\{\!\{\Delta\}\!\}$, and U' matches $U\lfloor^{lvl}$.

Because $(A^+\ \mathit{lvl})\lfloor^{lax}$ and $U\lfloor^{lvl}$, we can conclude that $U\lfloor^{lax}$.

Ξ matches $(\Theta{\circ}\Theta_B)\{\!\{\bigcirc B^+\}\!\}$ and $(\Theta{\circ}\Theta_B)\{B^+\}$ matches $\Theta\{\!\{\Theta_B\{B^+\}\}\!\}$. By the induction hypothesis on N and M we have $\Psi; (\Theta{\circ}\Theta_B)\{B^+\} \vdash U$, and the result follows by rule $\bigcirc_L$.

- $[\![V^< \mathit{Sp}]\!]^{A^+} M = V^<([\![\mathit{Sp}]\!]^{A^+} M)$
- $[\![V^> \mathit{Sp}]\!]^{A^+} M = V^>([\![\mathit{Sp}]\!]^{A^+} M)$

We must show $\Psi; \Xi \vdash U$, where

- Δ matches $\Theta_B\{\!\{[B_1^+ \twoheadrightarrow B_2^-], \Delta_B\}\!\}$, V is a derivation of $\Psi; \Delta_B \vdash [B_1^+]$,
  Sp is a derivation of $\Psi; \Theta_B\{[B_2^-]\} \vdash A^+\ \mathit{lvl}$,
- M is a derivation of $\Psi; \Theta\{A^+\} \vdash U$,
- Ξ matches $\Theta\{\!\{\Delta\}\!\}$, and $U\lfloor^{lvl}$.

Ξ matches $(\Theta{\circ}\Theta_B)\{\!\{[B_1^+ \twoheadrightarrow B_2^-], \Delta_B\}\!\}$ and $(\Theta{\circ}\Theta_B)\{[B_2^-]\}$ matches $\Theta\{\!\{\Theta_B\{[B_2^-]\}\}\!\}$. By invoking the induction hypothesis to substitute M out of Sp, we have $[\![\mathit{Sp}]\!]^{A^+} M$, which is a derivation of $\Psi; (\Theta{\circ}\Theta_B)\{[B_2^-]\} \vdash U$. The required result follows by rule $\twoheadrightarrow_L$ on V and $[\![\mathit{Sp}]\!]^{A^+} M$.

- $[\![\pi_1; \mathit{Sp}]\!]^{A^+} M = \pi_1; ([\![\mathit{Sp}]\!]^{A^+} M)$
- $[\![\pi_2; \mathit{Sp}]\!]^{A^+} M = \pi_2; ([\![\mathit{Sp}]\!]^{A^+} M)$
- $[\![[t]; \mathit{Sp}]\!]^{A^+} M = [t]; ([\![\mathit{Sp}]\!]^{A^+} M)$

Rightist substitution

Rightist substitutions encompass all the *right commutative cuts* from Pfenning's structural cut admissibility proof. The constructive content of this part is a function $[\![M/x]\!]^{A^-} E$; we say we are *substituting M into E*. Induction is on the second subterm, as we crawl through E looking for places where x is mentioned. The admissible rule associated with rightist substitution is rcut:

$$\frac{\Psi; \Delta \vdash M : A^- \quad \Psi; \Theta\{x{:}A^-\ \mathit{lvl}\} \vdash E : U \quad \Delta\ \mathit{stable}_L}{\Psi; \Theta\{\!\{\Delta\upharpoonleft_{\mathit{lvl}}\}\!\} \vdash [\![M/x]\!]^{A^-} E : U}\ \mathit{rcut}$$

A unique aspect of the right commutative cuts is that the implicit bookkeeping on contexts matters to the computational behavior of the proof: when we deal with multiplicative connectives like $A^+ \bullet B^+$ and $A^+ \rightarrowtail B^+$ under focus, we actually must consider that the variable x that we're substituting for can end up in only one specific branch of the proof (if x is associated with a judgment $A^-\ \mathit{ord}$ or $A^-\ \mathit{eph}$) or in both branches of the proof (if x is associated with a judgment $x{:}A^-\ \mathit{pers}$). The computational representation of these cases looks nondeterministic, but it is actually determined by the annotations and bookkeeping that we don't write down as part of the proof term. This is a point that we return to in Section 3.8.

For cases involving left rules, the general pattern is that Ξ matches $\Theta\{\!\{\Delta\restriction_{lvl}\}\!\}$ and the action of the left rule, when we read it bottom-up, is observe that $\Theta\{x{:}A^- \, lvl\}$ matches $\Theta'\{\!\{y{:}T \, ord\}\!\}$ in its conclusion and constructs $\Theta'\{\!\{y{:}T' \, lvl'\}\!\}$ in its premise(s). Effectively, we need to abstract a *two*-hole function (call it Γ) from Ξ. One hole – the place where x is – is defined by the frame Θ: morally, $\Theta = \lambda\Delta_B.\Gamma(x{:}A^- \, lvl)(\Delta_B)$. The other hole – the place where y is – is defined by Θ': morally, $\Theta' = \lambda\Delta_A.\Gamma(\Delta_A)(y{:}T \, ord)$. However, we cannot directly represent these functions due to the need to operate around matching constructs. Instead, we construct Θ_Δ to represent the frame that is morally $\lambda\Delta_B.\Gamma(\Delta)(\Delta_B)$, and $\Theta_{T'}$ to represent the frame that is morally $\lambda\Delta_A.\Gamma(\Delta_A)(y{:}T' \, lvl')$. As before, in cases that we discuss in detail, necessary properties of these two frames are stated but not proven.

Substitution into values

- $[\![M/x]\!]^{A^-} z = z$
- $[\![M/x]\!]^{A^-} (\downarrow N) = \downarrow([\![M/x]\!]^{A^-} N)$
- $[\![M/x]\!]^{A^-} (¡N) = ¡([\![M/x]\!]^{A^-} N)$
- $[\![M/x]\!]^{A^-} (!N) = !([\![M/x]\!]^{A^-} N)$

  We must show $\Psi; \Xi \vdash [!B^-]$, where

  - M is a derivation of $\Psi; \Delta \vdash A^-$,
  - $\Theta\{x{:}A^- \, lvl\}$ matches $\Delta'\restriction_{pers}$, N is a derivation of $\Psi; \Delta' \vdash B^-$,
  - and Ξ matches $\Theta\{\!\{\Delta\restriction_{lvl}\}\!\}$.

  Because $\Theta\{x{:}A^- \, lvl\}$ matches $\Delta'\restriction_{pers}$ and Ξ matches $\Theta\{\!\{\Delta\restriction_{lvl}\}\!\}$, we can conclude that there exists a Θ' such that $\Delta' = \Theta'\{x{:}A^- \, lvl\}$ and also that Ξ matches $\Xi\restriction_{pers}$.

  By the induction hypothesis on M and N, we have a derivation of $\Psi; \Xi \vdash B^-$, and the result follows by rule $!_R$.

- $[\![M/x]\!]^{A^-} () = ()$

  We must show $\Psi; \Xi \vdash [\mathbf{1}]$, where

  - M is a derivation of $\Psi; \Delta \vdash A^-$,
  - $\Theta\{x{:}A^- \, lvl\}$ matches $\cdot$,
  - and Ξ matches $\Theta\{\!\{\Delta\restriction_{lvl}\}\!\}$.

  Because $\Theta\{x{:}A^- \, lvl\}$ matches $\cdot$, it must be the case that $lvl = pers$, and so Ξ matches $\cdot$ as well. The result follows by rule $\mathbf{1}_R$.

- $[\![M/x]\!]^{A^-} (V_1 \bullet V_2) =$

$([\![M/x]\!]^{A^-} V_1) \bullet V_2$	*(if x is in V_1's context but not V_2's)*
$V_1 \bullet ([\![M/x]\!]^{A^-} V_2)$	*(if x is in V_2's context but not V_1's)*
$([\![M/x]\!]^{A^-} V_1) \bullet ([\![M/x]\!]^{A^-} V_2)$	*(if x is in both V_1 and V_2's contexts)*

  We must show $\Psi; \Xi \vdash [B_1^+ \bullet B_2^+]$, where

  - M is a derivation of $\Psi; \Delta \vdash A^-$,

- $\Theta\{x{:}A^- \, \mathit{lvl}\}$ matches Δ_1, Δ_2, V_1 is a derivation of $\Psi; \Delta_1 \vdash B_1^+$, V_2 is a derivation of $\Psi; \Delta_2 \vdash B_2^+$,
- and Ξ matches $\Theta\{\!\{\Delta\restriction_{\mathit{lvl}}\}\!\}$.

There are three possibilities: either x is a variable declaration in Δ_1 or Δ_2 but not both (if lvl is eph or ord) or x is a variable declaration in both Δ_1 and Δ_2 (if lvl is pers).

The first two cases are symmetric; assume without loss of generality that x is a variable declaration in Δ_1 but not Δ_2; we can construct a Θ_1 and Δ_1' such that $\Theta_1\{x{:}A^- \, \mathit{lvl}\} = \Delta_1$, Δ_1' matches $\Theta_1\{\!\{\Delta_1\restriction_{\mathit{lvl}}\}\!\}$, and Ξ matches Δ_1', Δ_2 By the induction hypothesis on M and V_1, we have $[\![M/x]\!]^{A^-} V_1$, a derivation of $\Psi; \Delta_1' \vdash [B_1^+]$, and the result follows by rule $\bullet_R$ on $[\![M/x]\!]^{A^-} V_1$ and V_2.

The third case is similar; we construct a Θ_1, Δ_1', Θ_2, and Δ_2' such that $\Theta_1\{x{:}A^- \, \mathit{lvl}\} = \Delta_1$, $\Theta_2\{x{:}A^- \, \mathit{lvl}\} = \Delta_2$, Δ_1' matches $\Theta_1\{\!\{\Delta_1\restriction_{\mathit{lvl}}\}\!\}$, Δ_2' matches $\Theta_1\{\!\{\Delta_2\restriction_{\mathit{lvl}}\}\!\}$, and Ξ matches Δ_1', Δ_2', which is only possible because $\mathit{lvl} = \mathit{pers}$; we then invoke the induction hypothesis twice.

– $[\![M/x]\!]^{A^-}(\text{INL}(V)) = \text{INL}([\![M/x]\!]^{A^-} V)$
– $[\![M/x]\!]^{A^-}(\text{INR}(V)) = \text{INR}([\![M/x]\!]^{A^-} V)$
– $[\![M/x]\!]^{A^-}(t, V) = t, ([\![M/x]\!]^{A^-} V)$
– $[\![M/x]\!]^{A^-}\text{REFL} = \text{REFL}$

Substitution into terms

– $[\![M/x]\!]^{A^-} V = [\![M/x]\!]^{A^-} V$
– $[\![M/x]\!]^{A^-}(y \cdot Sp) = y \cdot ([\![M/x]\!]^{A^-} Sp)$ *($x \# y$)*
– $[\![M/x]\!]^{A^-}(x \cdot Sp) =$
$(M \circ Sp)^{A^-}$ *(if x is not in Sp's context)*
$(M \circ ([\![M/x]\!]^{A^-} Sp))^{A^-}$ *(if x is in Sp's context)*

We must show $\Psi; \Xi \vdash U$, where

- M is a derivation of $\Psi; \Delta \vdash A^-$,
- $\Theta\{x{:}A^- \, \mathit{lvl}\}$ matches $\Theta'\{\!\{x{:}A^-\}\!\}$, Sp is a derivation of $\Psi; \Theta'\{[A^-]\} \vdash U$,
- and Ξ matches $\Theta\{\!\{\Delta\restriction_{\mathit{lvl}}\}\!\}$.

If lvl is eph or ord, then Ξ matches $\Theta'\{\!\{\Delta\}\!\}$, and the result follows by part 1 of cut admissibility on M and Sp.

If lvl is pers, Ξ doesn't match $\Theta'\{\!\{\Delta\}\!\}$, as Θ' has an extra variable declaration $x{:}A^- \, \mathit{pers}$. Instead, we know $\Theta\{[A^-]\}$ matches $\Theta_{[A^-]}\{\!\{\Delta\restriction_{\mathit{pers}}\}\!\}$ and $\Theta_{[A^-]}\{x{:}A^- \, \mathit{pers}\} = \Theta'\{[A^-]\}$, so Sp is also a derivation of $\Psi; \Theta_{[A^-]}\{x{:}A^- \, \mathit{pers}\} \vdash U$. By the induction hypothesis on M and Sp, we have $[\![M/x]\!]^{A^-} Sp$, a derivation of $\Psi; \Theta\{[A^-]\} \vdash U$. Then, because Ξ matches $\Theta\{\!\{\Delta\}\!\}$, the result follows from part 1 of cut admissibility on M and $[\![M/x]\!]^{A^-} Sp$.

– $[\![M/x]\!]^{A^-}(\langle z\rangle.N) = \langle z\rangle.([\![M/x]\!]^{A^-} N)$
– $[\![M/x]\!]^{A^-}\langle N\rangle = \langle [\![M/x]\!]^{A^-} N\rangle$

– $[\![M/x]\!]^{A^-}(\downarrow y.N) = \downarrow y.([\![M/x]\!]^{A^-} N)$
– $[\![M/x]\!]^{A^-}(¡y.N) = ¡y.([\![M/x]\!]^{A^-} N)$
– $[\![M/x]\!]^{A^-}(!y.N) = !y.([\![M/x]\!]^{A^-} N)$

We must show $\Psi; \Xi \vdash U$, where

- M is a derivation of $\Psi; \Delta \vdash A^-$,
- $\Theta\{x{:}A^-\ \mathit{lvl}\}$ matches $\Theta'\{\!\{!B^-\}\!\}$, N is a derivation of $\Psi; \Theta'\{y{:}B^-\ \mathit{pers}\} \vdash U$,
- and Ξ matches $\Theta\{\!\{\Delta\restriction_{\mathit{lvl}}\}\!\}$.

Let $\Delta' = \Delta, y{:}B^-\ \mathit{pers}$. By admissible weakening, M is derivation of $\Psi; \Delta' \vdash A^-$ too. Ξ matches $\Theta_\Delta\{\!\{!B^-\}\!\}$, $\Theta_\Delta\{y{:}B^-\ \mathit{pers}\}$ matches $\Theta_{B^-}\{\!\{\Delta'\restriction_{\mathit{lvl}}\}\!\}$, and $\Theta_{B^-}\{x{:}A^-\ \mathit{lvl}\} = \Theta'\{y{:}B^-\ \mathit{pers}\}$. By the induction hypothesis on M and N we have $\Psi; \Theta_\Delta\{y{:}B^-\ \mathit{pers}\} \vdash U$, and the result follows by rule $!_L$.

– $[\![M/x]\!]^{A^-}(\uparrow N) = \uparrow([\![M/x]\!]^{A^-} N)$
– $[\![M/x]\!]^{A^-}\{N\} = \{[\![M/x]\!]^{A^-} N\}$
– $[\![M/x]\!]^{A^-} \bullet N = \bullet([\![M/x]\!]^{A^-} N)$
– $[\![M/x]\!]^{A^-} \lambda^< N = \lambda^<([\![M/x]\!]^{A^-} N)$
– $[\![M/x]\!]^{A^-} \lambda^> N = \lambda^>([\![M/x]\!]^{A^-} N)$
– $[\![M/x]\!]^{A^-} \text{ABORT} = \text{ABORT}$
– $[\![M/x]\!]^{A^-}[N_1, N_2] = [[\![M/x]\!]^{A^-} N_1, [\![M/x]\!]^{A^-} N_2]$

We must show $\Psi; \Xi \vdash U$, where

- M is a derivation of $\Psi; \Delta \vdash A^-$,
- $\Theta\{x{:}A^-\ \mathit{lvl}\}$ matches $\Theta'\{\!\{B_1^+ \oplus B_2^+\}\!\}$, N_1 is a derivation of $\Psi; \Theta'\{B_1^+\} \vdash U$, N_2 is a derivation of $\Psi; \Theta'\{B_2^+\} \vdash U$,
- and Ξ matches $\Theta\{\!\{\Delta\restriction_{\mathit{lvl}}\}\!\}$.

Ξ matches $\Theta_\Delta\{\!\{B_1^+ \oplus B_2^+\}\!\}$, and for $i \in \{1,2\}$, $\Theta_\Delta\{B_i^+\}$ matches $\Theta_{B_i^+}\{\!\{\Delta\restriction_{\mathit{pers}}\}\!\}$ and $\Theta_{B_i^+}\{x{:}A^-\ \mathit{lvl}\} = \Theta'\{B_i^+\}$.

By the induction hypothesis on M and N_1, we have $\Psi; \Theta_\Delta\{B_1^+\} \vdash U$, by the induction hypothesis on M and N_2, we have $\Psi; \Theta_\Delta\{B_2^+\} \vdash U$, and the result follows by rule $\oplus_L$.

– $[\![M/x]\!]^{A^-} \top = \top$
– $[\![M/x]\!]^{A^-}(N_1 \,\&\, N_2) = ([\![M/x]\!]^{A^-} N_1) \,\&\, ([\![M/x]\!]^{A^-} N_2)$
– $[\![M/x]\!]^{A^-} a.N = a.([\![M/x]\!]^{A^-} N)$
– $[\![M/x]\!]^{A^-}[a].N = [a].([\![M/x]\!]^{A^-} N)$
– $[\![M/x]\!]^{A^-} \text{UNIF}\,(\mathsf{fn}\,\sigma \Rightarrow \phi(\sigma)) = \text{UNIF}\,(\mathsf{fn}\,\sigma \Rightarrow [\![\sigma M/x]\!]^{A^-} \phi(\sigma))$

We must show $\Psi; \Xi \vdash U$, where

- M is a derivation of $\Psi; \Delta \vdash A^-$,
- $\Theta\{x{:}A^-\ \mathit{lvl}\}$ matches $\Theta'\{\!\{t \doteq s\}\!\}$, ϕ is a function from substitutions $\Psi' \vdash \sigma : \Psi$ that

unify t and s to derivations of $\Psi'; \sigma\Theta'\{\cdot\} \vdash \sigma U$.

- and Ξ matches $\Theta\{\!\{\Delta\restriction_{lvl}\}\!\}$.

Ξ matches $\Theta_\Delta\{\!\{t \doteq s\}\!\}$, and for any substitution σ, $\sigma\Theta_\Delta\{\cdot\}$ matches $\sigma\Theta\{\!\{\Delta\restriction_{lvl}\}\!\}$. By rule $\doteq_L$, it suffices to show that, given an arbitrary substitution $\Psi' \vdash \sigma : \Psi$, there is a derivation of $\Psi'; \sigma\Theta_\Delta\{\cdot\} \vdash \sigma U$.

By applying σ to ϕ, we get $\phi(\sigma)$, a derivation of $\Psi'; \sigma\Theta_B\{\cdot\} \vdash \sigma A^+$; the usual interpretation of higher-order derivations is that $\phi(\sigma)$ is a subderivation of ϕ, so $\phi(\sigma)$ can be used to invoke the induction hypothesis. From variable substitution, we get σM, a derivation of $\Psi'; \sigma\Delta \vdash \sigma A^-\ lvl$, and the result follows by the induction hypothesis on σM and $\phi(\sigma)$.

Substitution into spines

– $[\![M/x]\!]^{A^-}\textsc{nil} = \textsc{nil}$

– $[\![M/x]\!]^{A^-}(\uparrow N) = \uparrow([\![M/x]\!]^{A^-} N)$

– $[\![M/x]\!]^{A^-}\{N\} = \{[\![M/x]\!]^{A^-} N\}$

– $[\![M/x]\!]^{A^-} V^{<} Sp =$

$([\![M/x]\!]^{A^-} V)^{<} Sp$	*(if x is in V's context but not Sp's)*
$V^{<}([\![M/x]\!]^{A^-} Sp)$	*(if x is in Sp's context but not V's)*
$([\![M/x]\!]^{A^-} V)^{<}([\![M/x]\!]^{A^-} Sp)$	*(if x is in both V and Sp's contexts)*

– $[\![M/x]\!]^{A^-} V^{>} Sp =$

$([\![M/x]\!]^{A^-} V)^{>} Sp$	*(if x is in V's context but not Sp's)*
$V^{>}([\![M/x]\!]^{A^-} Sp)$	*(if x is in Sp's context but not V's)*
$([\![M/x]\!]^{A^-} V)^{>}([\![M/x]\!]^{A^-} Sp)$	*(if x is in both V and Sp's contexts)*

We must show $\Psi; \Xi \vdash \forall x{:}\tau.B^-$, where

- M is a derivation of $\Psi; \Delta \vdash A^-$,
- $\Theta\{x{:}A^-\ lvl\}$ matches $\Theta'\{\!\{[B_1^+ \twoheadrightarrow B_2^-], \Delta_A\}\!\}$, V is a derivation of $\Psi; \Delta_A \vdash [B_1^+]$, Sp is a derivation of $\Psi; \Theta'\{[B_2^-]\} \vdash U$,
- and Ξ matches $\Theta\{\!\{\Delta\restriction_{lvl}\}\!\}$.

There are three possibilities: either x is a variable declaration in Θ' or Δ_A but not both (if lvl is eph or ord) or x is a variable declaration in both Θ' and Δ_A (if lvl is $pers$).

In the first case (x is a variable declaration in Δ_A only), Ξ matches $\Theta'\{\!\{[B_1^+ \twoheadrightarrow B_2^-], \Delta'_A\}\!\}$, Δ'_A matches $\Theta_A\{\!\{\Delta\restriction_{lvl}\}\!\}$, and $\Delta_A = \Theta_A\{x{:}A^-\ lvl\}$. By the induction hypothesis on M and V we have $[\![M/x]\!]^{A^-} V$, a derivation of $\Psi; \Delta'_A \vdash [B_1^+]$, and the result follows by rule $\twoheadrightarrow_L$ on $[\![M/x]\!]^{A^-} V$ and Sp.

In the second case (x is in Θ' only), Ξ matches $\Theta_\Delta\{\!\{[B_1^+ \twoheadrightarrow B_2^-], \Delta_A\}\!\}$, $\Theta_\Delta\{[B_2^-]\}$ matches $\Theta_{[B_2^-]}\{\!\{\Delta\restriction_{lvl}\}\!\}$, and $\Theta_{[B_2^-]}\{x{:}lvl\} = \Theta'\{[B_2^-]\}$. By the induction hypothesis on M and Sp, we have $[\![M/x]\!]^{A^-} Sp$, a derivation of $\Psi; \Theta_\Delta\{[B_2^-]\} \vdash U$, and the result follows by rule $\twoheadrightarrow_L$ on V and $[\![M/x]\!]^{A^-} Sp$.

In the third case (x is in Θ' and Δ_A), Ξ matches $\Theta_\Delta\{\!\{[B_1^+ \twoheadrightarrow B_2^-], \Delta'_A\}\!\}$, where Θ_Δ and Δ'_A have the same properties as before, and we proceed invoking the induction hypothesis twice.

– $[\![M/x]\!]^{A^-} \pi_1; Sp = \pi_1; ([\![M/x]\!]^{A^-} Sp)$
– $[\![M/x]\!]^{A^-} \pi_2; Sp = \pi_2; ([\![M/x]\!]^{A^-} Sp)$
– $[\![M/x]\!]^{A^-} [t]; Sp = [t]; ([\![M/x]\!]^{A^-} Sp)$

3.5 Identity expansion

The form of the identity expansion theorems is already available to us: the admissible rules $\eta_{A^+_{lvl}}$ and $\eta_{A^-_{lvl}}$ are straightforward generalizations of the explicit rules η^+ and η^- in Figure 3.5 from ordered atomic propositions p^+ and p^- to arbitrary propositions and from permeable atomic propositions p^+_{eph}, p^+_{pers}, and p^-_{lax} to arbitrary permeable propositions A^+_{eph}, A^+_{pers} and A^-_{lax}. The content of Theorem 3.7 below is captured by the two admissible rules $\eta_{A^+_{lvl}}$ and $\eta_{A^-_{lvl}}$ and also by the two functions and $\eta_{A^+_{lvl}}(z.N)$ and $\eta_{A^-_{lvl}}(N)$ that operate on proof terms.

$$\frac{\Psi; \Theta\{z{:}\langle A^+_{lvl}\rangle\ lvl\} \vdash N : U}{\Psi; \Theta\{\!\{A^+_{lvl}\}\!\} \vdash \eta_{A^+_{lvl}}(z.N) : U}\ \eta_{A^+_{lvl}} \qquad \frac{\Psi; \Delta \vdash N : \langle A^-_{lvl}\rangle\ lvl \quad \Delta\ stable_L}{\Psi; \Delta \vdash \eta_{A^-_{lvl}}(N) : A^-_{lvl}}\ \eta_{A^-_{lvl}}$$

Identity expansion is not the perhaps not the best name for the property; the name comes from the fact that the usual identity properties are a corollary of identity expansion. Specifically, $\eta_{A^+}(z.z)$ is a derivation of $\Psi; A^+ \vdash A^+\ true$ and $\eta_{A^-}(x \cdot \text{NIL})$ is a derivation of $\Psi; x{:}A^-\ ord \vdash A^-$.

In the proof of identity expansion, we do pay some price in return for including permeable propositions, as we perform slightly different bookkeeping depending on whether or not it is necessary to apply admissible weakening to the subderivation N. However, this cost is mostly borne by the part of the context we leave implicit.

Theorem 3.7 (Identity expansion)**.**
∗ *If* $\Psi; \Theta\{z{:}\langle A^+_{lvl}\rangle\ lvl\} \vdash U$ *and* Δ *matches* $\Theta\{\!\{A^+_{lvl}\}\!\}$*, then* $\Psi; \Delta \vdash U$.
∗ *If* $\Psi; \Delta \vdash \langle A^-_{lvl}\rangle\ lvl$ *and* Δ *is stable, then* $\Psi; \Delta \vdash A^-_{lvl}$.

Proof. By mutual induction over the structure of types. We provide the full definition at the level of proof terms and include an extra explanatory derivation for a few of the positive cases.

Positive cases

– $\eta_{p^+_{lvl}}(z.N) = \langle z\rangle.N$
N is a derivation of $\Psi; \Theta\{z{:}p^+_{lvl}\ lvl\} \vdash U$; the result follows immediately by the rule η^+:

$$\frac{\Psi; \Theta\{z{:}p^+_{lvl}\ lvl\} \vdash N : U}{\Psi; \Theta\{\!\{p^+_{lvl}\}\!\} \vdash \langle z\rangle.N : U}\ \eta^+$$

– $\eta_{\downarrow A^-}(z.N) = {\downarrow}x.([({\downarrow}(\eta_{A^-}(x \cdot \textsc{nil})))/z]N)$
N is a derivation of $\Psi; \Theta\{z{:}\langle {\downarrow}A^- \rangle\ \mathit{ord}\} \vdash U$. We construct a context Ξ that contains only the persistent propositions from Δ. This means $\Theta\{\!\{\Xi, x{:}A^-\ \mathit{ord}\}\!\}$ matches $\Theta\{x{:}A^-\ \mathit{ord}\}$. We can then derive:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Psi; \Xi, [A^-] \vdash \textsc{nil} : \langle A^- \rangle\ \mathit{true}}\ \mathit{id}^-}{\Psi; \Xi, x{:}A^-\ \mathit{ord} \vdash x \cdot \textsc{nil} : \langle A^- \rangle\ \mathit{true}}\ \mathit{focus}_L}{\Psi; \Xi, x{:}A^-\ \mathit{ord} \vdash \eta_{A^-}(x \cdot \textsc{nil}) : A^-\ \mathit{true}}\ \eta_{A^-}}{\Psi; \Xi, x{:}A^-\ \mathit{ord} \vdash {\downarrow}(\eta_{A^-}(x \cdot \textsc{nil})) : [{\downarrow}A^-]}\ {\downarrow}_R \quad \Psi; \Theta\{z{:}\langle {\downarrow}A^- \rangle\ \mathit{ord}\} \vdash N : U}{\dfrac{\Psi; \Theta\{x{:}A^-\ \mathit{ord}\} \vdash [({\downarrow}(\eta_{A^-}(x \cdot \textsc{nil})))/z]N : U}{\Psi; \Theta\{\!\{{\downarrow}A^-\}\!\} \vdash {\downarrow}x.([({\downarrow}(\eta_{A^-}(x \cdot \textsc{nil})))/z]N) : U}\ {\downarrow}_L}\ \mathit{subst}^+$$

– $\eta_{¡A^-}(z.N) = ¡x.([(¡(\eta_{A^-}(x \cdot \textsc{nil})))/z]N)$
– $\eta_{!A^-}(z.N) = !x.([(!(\eta_{A^-}(x \cdot \textsc{nil})))/z]N)$
N is a derivation of $\Psi; \Theta\{z{:}\langle !A^- \rangle\ \mathit{lvl}\} \vdash U$, where lvl can be anything (ord, eph, or pers). We construct a context Ξ that contains only the persistent propositions from Δ and a frame Θ^+ that is Θ plus an extra variable declaration $x{:}A^-\ \mathit{pers}$. This means that $\Theta\{x{:}A^-\ \mathit{pers}\}$ matches $\Theta^+\{\!\{(\Xi, x{:}A^-\ \mathit{pers})\restriction_{\mathit{lvl}}\}\!\}$. We can then derive:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Psi; \Xi, x{:}A^-\ \mathit{pers}, [A^-] \vdash \textsc{nil} : \langle A^- \rangle\ \mathit{true}}\ \mathit{id}^-}{\Psi; \Xi, x{:}A^-\ \mathit{pers} \vdash x \cdot \textsc{nil} : \langle A^- \rangle\ \mathit{true}}\ \mathit{focus}_L}{\Psi; \Xi, x{:}A^-\ \mathit{pers} \vdash \eta_{A^-}(x \cdot \textsc{nil}) : A^-\ \mathit{true}}\ \eta_{A^-}}{\Psi; \Xi, x{:}A^-\ \mathit{pers} \vdash !(\eta_{A^-}(x \cdot \textsc{nil})) : [!A^-]}\ !_R \quad \dfrac{\Psi; \Theta\{z{:}\langle {\downarrow}A^- \rangle\ \mathit{lvl}\} \vdash N : U}{\Psi; \Theta^+\{z{:}\langle {\downarrow}A^- \rangle\ \mathit{lvl}\} \vdash N : U}\ \mathit{weaken}}{\dfrac{\Psi; \Theta\{x{:}A^-\ \mathit{pers}\} \vdash [(!(\eta_{A^-}(x \cdot \textsc{nil})))/z]N : U}{\Psi; \Theta\{\!\{!A^-\}\!\} \vdash !x.([(!(\eta_{A^-}(x \cdot \textsc{nil})))/z]N) : U}\ !_L}\ \mathit{subst}^+$$

– $\eta_{\mathbf{1}}(z.N) = ().([()/z]N)$
– $\eta_{A^+_{\mathit{lvl}} \bullet B^+_{\mathit{lvl}}}(z.N) = \bullet\big(\eta_{A^+_{\mathit{lvl}}}(z_1.\ \eta_{B^+_{\mathit{lvl}}}(z_2.\ [(z_1 \bullet z_2)/z]N))\big)$
N is a derivation of $\Psi; \Theta\{z{:}\langle A^+_{\mathit{lvl}} \bullet B^+_{\mathit{lvl}} \rangle\ \mathit{lvl}\} \vdash U$, where lvl can be anything (ord, eph, or pers). We construct a context Ξ that contains only the persistent propositions from Δ and a frame Θ^+ that is either Θ (if lvl is ord or eph) or it is Θ plus additional variable declarations $z_1{:}\langle A^+_{\mathit{lvl}} \rangle\ \mathit{lvl}$ and $z_2{:}\langle B^+_{\mathit{lvl}} \rangle\ \mathit{lvl}$ (if lvl is pers). This means that $\Theta\{x_1{:}\langle A^+_{\mathit{lvl}} \rangle\ \mathit{lvl}, x_2{:}\langle B^+_{\mathit{lvl}} \rangle\ \mathit{lvl}\}$ matches $\Theta^+\{\!\{(\Xi, x_1{:}\langle A^+_{\mathit{lvl}} \rangle\ \mathit{lvl}, x_2{:}\langle B^+_{\mathit{lvl}} \rangle\ \mathit{lvl})\restriction_{\mathit{lvl}}\}\!\}$. We can then derive:

$$\dfrac{\dfrac{\dfrac{}{\Psi; \Xi_1 \vdash z_1 : [A^+_{\mathit{lvl}}]}\ \mathit{id}^+ \quad \dfrac{}{\Psi; \Xi_2 \vdash z_2 : [B^+_{\mathit{lvl}}]}\ \mathit{id}^+}{\Psi; \Xi, z_1{:}\langle A^+_{\mathit{lvl}} \rangle\ \mathit{lvl}, z_2{:}\langle B^+_{\mathit{lvl}} \rangle\ \mathit{lvl} \vdash z_1 \bullet z_2 : [A^+_{\mathit{lvl}} \bullet B^+_{\mathit{lvl}}]}\ \bullet_R \quad \dfrac{\Psi; \Theta\{z{:}\langle A^+_{\mathit{lvl}} \bullet B^+_{\mathit{lvl}} \rangle\ \mathit{lvl}\} \vdash N : U}{\Psi; \Theta^+\{z{:}\langle A^+_{\mathit{lvl}} \bullet B^+_{\mathit{lvl}} \rangle\ \mathit{lvl}\} \vdash N : U}\ \mathit{weaken}}{\dfrac{\dfrac{\dfrac{\Psi; \Theta\{z_1{:}\langle A^+_{\mathit{lvl}} \rangle\ \mathit{lvl}, z_2{:}\langle B^+_{\mathit{lvl}} \rangle\ \mathit{lvl}\} \vdash [(z_1 \bullet z_2)/z]N : U}{\Psi; \Theta\{z_1{:}\langle A^+_{\mathit{lvl}} \rangle\ \mathit{lvl}, B^+_{\mathit{lvl}}\} \vdash \eta_{B^+_{\mathit{lvl}}}(z_2.\ [(z_1 \bullet z_2)/z]N) : U}\ \eta_{B^+_{\mathit{lvl}}}}{\Psi; \Theta\{A^+_{\mathit{lvl}}, B^+_{\mathit{lvl}}\} \vdash \eta_{A^+_{\mathit{lvl}}}(z_1.\ \eta_{B^+_{\mathit{lvl}}}(z_2.\ [(z_1 \bullet z_2)/z]N)) : U}\ \eta_{A^+_{\mathit{lvl}}}}{\Psi; \Theta\{\!\{A^+_{\mathit{lvl}} \bullet B^+_{\mathit{lvl}}\}\!\} \vdash \bullet(\eta_{A^+_{\mathit{lvl}}}(z_1.\ \eta_{B^+_{\mathit{lvl}}}(z_2.\ [(z_1 \bullet z_2)/z]N))) : U}\ \bullet_L}\ \mathit{subst}^+$$

Either Ξ_1 and Ξ_2 are both $\Xi, z_1{:}\langle A^+_{\mathit{lvl}} \rangle\ \mathit{lvl}, z_2{:}\langle B^+_{\mathit{lvl}} \rangle\ \mathit{lvl}$ (if lvl is pers), or Ξ_1 is $\Xi, z_1{:}\langle A^+_{\mathit{lvl}} \rangle\ \mathit{lvl}$ and Ξ_2 is $\Xi, z_2{:}\langle B^+_{\mathit{lvl}} \rangle\ \mathit{lvl}$ (if lvl is ord or eph).

- $\eta_{\mathbf{0}}(z.N) = \text{ABORT}$
- $\eta_{A^+_{lvl} \oplus B^+_{lvl}}(z.N) = [\eta_{A^+_{lvl}}(z_1.\ [\text{INL}(z_1)/z]N), \eta_{B^+_{lvl}}(z_2.\ [\text{INR}(z_2)/z]N)]$
- $\eta_{\exists a:\tau.A^+_{lvl}}(z.N) = a.\eta_{A^+}(z'.\ [(a, z')/z]N)$
- $\eta_{t \doteq_\tau s}(z.N) = \text{UNIF}\,(\mathsf{fn}\ \sigma \Rightarrow [\text{REFL}/z](\sigma N))$

Negative cases

- $\eta_{p^-_{lvl}}(N) = \langle N \rangle$
- $\eta_{\uparrow A^+}(N) = \uparrow([N](\uparrow(\eta_{A^+}(z.z))))$
- $\eta_{\bigcirc A^+}(N) = \{[N](\{\eta_{A^+}(z.z)\})\}$
- $\eta_{A^+ \rightarrowtail B^-_{lvl}}(N) = \lambda^<(\eta_{A^+}(z.\ \eta_{B^-_{lvl}}([N](z^< \text{NIL}))))$
- $\eta_{A^+ \twoheadrightarrow B^-_{lvl}}(N) = \lambda^>(\eta_{A^+}(z.\ \eta_{B^-_{lvl}}([N](z^> \text{NIL}))))$
- $\eta_{\top}(N) = \top$
- $\eta_{A^-_{lvl} \& B^-_{lvl}}(N) = (\eta_{A^-_{lvl}}([N](\pi_1; \text{NIL}))) \mathbin{\&} (\eta_{B^-_{lvl}}([N](\pi_2; \text{NIL})))$
- $\eta_{\forall a:\tau.A^-_{lvl}}(N) = [a].(\eta_{A^-_{lvl}}([N]([a]; \text{NIL})))$

□

3.6 Correctness of focusing

Our proof of the correctness of focusing is based on erasure as described in Section 2.3.7. The argument follows the one from the structural focalization development, and the key component is the set of *unfocused admissibility lemmas*, lemmas that establish that each of the reasoning steps that can be made in unfocused OL_3 are admissible inferences made on stable sequents in focused OL_3.

3.6.1 Erasure

As in Section 2.3.7, we define erasure only on stable, suspension-normal sequents. Erasure for propositions is defined as in Figure 3.8. As discussed in Section 2.5.4, even though we have not incorporated a notion of permeable and mobile atomic propositions into the unfocused presentation of OL_3, it is possible to erase a permeable atomic proposition p^+_{pers} as $!p^+_{pers}$.[3] In this way, we can see the separation criteria from our previous work [SP08, PS09] arising as an emergent property of erasure.

We have to define erasure on non-stable sequents in order for the soundness of focusing to go through, though we will only define erasure on suspension-normal sequents. The erasure of sequents, U°, maps polarized succedents $A^+\ lvl$, $\langle p^-_{lvl} \rangle\ lvl$, $[A^+]$, and A^- in the obvious way to unpolarized succedents $(A^+)^\circ\ lvl$, $p^-_{lvl}\ lvl$, $(A^+)^\circ\ ord$, and $(A^-)^\circ\ ord$, respectively. To describe the erasure of contexts more simply, we will assume that we can give a presentation of unfocused

[3] The polarity and level annotations are meaningless in the unfocused logic. We keep them only to emphasize that p^+_{pers} and p^-_{lax} do *not* erase to the same unpolarized atomic proposition p but two distinct unpolarized atomic propositions.

$$
\begin{array}{rcl|rcl}
\multicolumn{3}{l|}{\boxed{(A^+)^\circ}} & \multicolumn{3}{l}{\boxed{(A^-)^\circ}} \\
(p^+)^\circ & = & p^+ & (p^-)^\circ & = & p^- \\
(p^+_{eph})^\circ & = & ¡p^+_{eph} & (p^-_{lax})^\circ & = & \bigcirc p^-_{lax} \\
(p^+_{pers})^\circ & = & !p^+_{pers} & & & \\
({\downarrow}A^-)^\circ & = & (A^-)^\circ & ({\uparrow}A^+)^\circ & = & (A^+)^\circ \\
(¡A^-)^\circ & = & ¡(A^-)^\circ & (\bigcirc A^+)^\circ & = & \bigcirc(A^+)^\circ \\
(!A^-)^\circ & = & !(A^-)^\circ & & & \\
(\mathbf{1})^\circ & = & \mathbf{1} & (A^+ \rightarrowtail B^-)^\circ & = & (A^+)^\circ \rightarrowtail (B^+)^\circ \\
(A^+ \bullet B^+)^\circ & = & (A^+)^\circ \bullet (B^+)^\circ & (A^+ \twoheadrightarrow B^-)^\circ & = & (A^+)^\circ \twoheadrightarrow (B^-)^\circ \\
(\mathbf{0})^\circ & = & \mathbf{0} & (\top)^\circ & = & \top \\
(A^+ \oplus B^+)^\circ & = & (A^+)^\circ \oplus (B^+)^\circ & (A^- \mathbin{\&} B^-)^\circ & = & (A^-)^\circ \mathbin{\&} (B^-)^\circ \\
(\exists a{:}\tau.A^+)^\circ & = & \exists a{:}\tau.(A^+)^\circ & (\forall a{:}\tau.A^-)^\circ & = & \forall a{:}\tau.(A^-)^\circ \\
(t \doteq s)^\circ & = & t \doteq s & & &
\end{array}
$$

Figure 3.8: Erasure in OL_3

OL_3 that uses unified substructural contexts, as we outlined in Section 3.2; the judgments of this presentation have the form $\Psi;\Delta \Longrightarrow U$. In this presentation, we can define Δ° that takes every variable declaration $x{:}A^-\ \mathit{lvl}$, $x{:}\langle p^+_{\mathit{lvl}}\rangle\ \mathit{lvl}$, $[A^-]$, or A^+ to a variable declaration $x{:}(A^-)^\circ\ \mathit{lvl}$, $x{:}p^+_{\mathit{lvl}}\ \mathit{lvl}$, $x{:}(A^-)^\circ\ \mathit{ord}$, or $x{:}(A^+)^\circ\ \mathit{ord}$ (and in the process, either comes up with or reveals the suppressed variable names associated with focused negative propositions and inverting positive propositions). Erasure of succedents U° is similar: $(A^+\ \mathit{lvl})^\circ = (A^+)^\circ\ \mathit{lvl}$, $(\langle p^-_{\mathit{lvl}}\rangle\ \mathit{lvl})^\circ = p^-_{\mathit{lvl}}\ \mathit{lvl}$, $([A^-])^\circ = (A^-)^\circ\ \mathit{true}$, and $(A^+)^\circ = (A^+)^\circ\ \mathit{true}$.

3.6.2 De-focalization

The act of taking a focused proof of a sequent and getting an unfocused proof of the corresponding erased sequent is *de-focalization*. If we run the constructive content of the proof of the soundness of focusing (the OL_3 analogue of Theorem 2.5 from Section 2.3.7), the proof performs de-focalization.

Theorem 3.8 (Soundness of focusing/de-focalization)**.** *If* $\Psi;\Delta \vdash U$*, then* $\Psi;\Delta^\circ \Longrightarrow U^\circ$.

Proof. By induction over the structure of focused proofs. Most rules ($\bullet_L$, $\twoheadrightarrow_R$, etc.) in the focused derivations have an obviously analogous rule in the unfocused logic, and for the four rules dealing with shifts, the necessary result follows directly from the induction hypothesis. The focus_L rule potentially requires an instance of the admissible copy or place rules in unfocused OL_3, and the focus_R rule potentially requires an instance of the admissible lax rule in unfocused OL_3. □

3.6.3 Unfocused admissibility

Unfocused admissibility has a structure that is unchanged from the previous discussion in the proof of the completeness of focusing for linear logic (Theorem 2.6 in Section 2.3.7). In this

Atomic propositions

$$\frac{}{\Psi; z{:}\langle p^+_{lvl}\rangle \vdash z : p^+_{lvl}\ \mathit{lvl}'} \qquad \frac{}{\Psi; x{:}\uparrow p^+_{lvl} \vdash x \cdot \uparrow\langle z\rangle.\ z : p^+_{lvl}\ \mathit{lvl}'}$$

$$\frac{}{\Psi; x{:}p^-_{lvl} \vdash x \cdot \textsc{nil} : \langle p^-_{lvl}\rangle\ \mathit{lvl}} \qquad \frac{}{\Psi; x{:}p^-_{lvl} \vdash \downarrow\langle x \cdot \textsc{nil}\rangle : \downarrow p^-_{lvl}\ \mathit{lvl}}$$

Exponentials

$$\frac{\Psi; \Delta \vdash N : A^+\ \mathit{ord}}{\Psi; \Delta \vdash \downarrow\uparrow N : \downarrow\uparrow A^+\ \mathit{lvl}} \qquad \frac{\Psi; \Theta\{x'{:}A^-\ \mathit{ord}\} \vdash N : U}{\Psi; \Theta\{\!\{x{:}\uparrow\downarrow A^-\}\!\} \vdash x \cdot \uparrow\downarrow x'.N : U}$$

$$\frac{\Psi; \Delta \vdash N : \downarrow A^-\ \mathit{ord}}{\Psi; \Delta\upharpoonright_{\mathit{eph}} \vdash [\![\uparrow N/x]\!]^{\uparrow\downarrow A^-}\ \text{¡}\eta_{A^-}(x \cdot \uparrow(\downarrow x'.\ (x' \cdot \textsc{nil}))) : \text{¡}A^-\ \mathit{lvl}} \qquad \frac{\Psi; \Theta\{x'{:}A^-\ \mathit{eph}\} \vdash N : U}{\Psi; \Theta\{\!\{x{:}\uparrow\text{¡}A^-\}\!\} \vdash x \cdot \uparrow\text{¡}x'.N : U}$$

$$\frac{\Psi; \Delta \vdash N : \downarrow A^-\ \mathit{ord}}{\Psi; \Delta\upharpoonright_{\mathit{pers}} \vdash [\![\uparrow N/x]\!]^{\uparrow\downarrow A^-}\ !\eta_{A^-}(x \cdot \uparrow(\downarrow x'.\ (x' \cdot \textsc{nil}))) : !A^-\ \mathit{lvl}} \qquad \frac{\Psi; \Theta\{x'{:}A^-\ \mathit{pers}\} \vdash N : U}{\Psi; \Theta\{\!\{x{:}\uparrow !A^-\}\!\} \vdash x \cdot \uparrow !x'.N : U}$$

$$\frac{\Psi; \Delta \vdash N : A^+\ \mathit{lax}}{\Psi; \Delta \vdash \downarrow\{N\} : \downarrow\bigcirc A^+\ \mathit{lvl}} \qquad \frac{\Psi; \Theta\{x'{:}\uparrow A^+\ \mathit{ord}\} \vdash N : U}{\Psi; \Theta\{\!\{x{:}\bigcirc A^+\}\!\} \vdash [\![x \cdot \eta_{A^+}(z.\ \downarrow\uparrow z)]\!]^{\downarrow\uparrow A^+} \downarrow x'.N : U\lfloor^{\mathit{lax}}}$$

Multiplicative connectives ($\rightarrowtail$ and $\twoheadrightarrow$ are symmetric)

$$\frac{}{\Psi; \cdot \vdash () : \mathbf{1}\ \mathit{lvl}} \qquad \frac{\Psi; \Theta\{\cdot\} \vdash N : U}{\Psi; \Theta\{\!\{x{:}\uparrow\mathbf{1}\}\!\} \vdash x \cdot \uparrow().N : U}$$

$$\frac{\Psi; \Delta_1 \vdash N_1 : A^+\ \mathit{ord} \quad \Psi; \Delta_2 \vdash N_2 : B^+\ \mathit{ord}}{\Psi; \Delta_1, \Delta_2 \vdash [\![\uparrow N_1/x_1]\!]^{\uparrow A^+}([\![N_2]\!]^{B^+}\eta_{B^+}(z_2.\ x_1 \cdot \uparrow\eta_{A^+}(z_1.\ z_1 \bullet z_2))) : A^+ \bullet B^+\ \mathit{lvl}}$$

$$\frac{\Psi; \Theta\{x_1{:}\uparrow A^+, x_2{:}\uparrow B^+\} \vdash N : U}{\Psi; \Theta\{\!\{x{:}\uparrow(A^+ \bullet B^+)\}\!\} \vdash x \cdot \uparrow[\![\bullet(\eta_{A^+}(z_1.\ \eta_{B^+}(z_2.\ \downarrow\uparrow z_1 \bullet \downarrow\uparrow z_2)))]\!]^{\downarrow\uparrow A^+ \bullet \downarrow\uparrow B^+} \bullet(\downarrow x_1.\downarrow x_2.N) : U}$$

$$\frac{\Psi; x{:}\uparrow A^+\ \mathit{ord}, \Delta \vdash N : \downarrow B^-\ \mathit{ord}}{\Psi; \Delta \vdash [\![(\lambda^<(\downarrow x.\uparrow N))/x']\!]^{\downarrow\uparrow A^+ \rightarrowtail \uparrow\downarrow B^-}(\downarrow\lambda^<\eta_{A^+}(z.\ \eta_{B^-}(x' \cdot (\downarrow\uparrow z)^<(\uparrow\downarrow x''.x'' \cdot \textsc{nil})))) : \downarrow(A^+ \rightarrowtail B^-)\ \mathit{ord}}$$

$$\frac{\Psi; \Delta_A \vdash N_1 : A^+\ \mathit{ord} \quad \Psi; \Theta\{x'{:}B^-\ \mathit{ord}\} \vdash N_2 : U}{\Psi; \{\!\{\Delta_A, x{:}A^+ \rightarrowtail B^-\}\!\} \vdash [\![[\![N_1]\!]\eta_{A^+}(z.\ \downarrow\eta_{B^-}(x \cdot z^<\textsc{nil}))]\!]\downarrow x'.N_2 : U}$$

Figure 3.9: Unfocused admissibility for the multiplicative, exponential fragment of OL_3

$$\frac{}{\Psi;\Theta\{\!\{x{:}{\uparrow}\mathbf{0}\}\!\} \vdash x \cdot {\uparrow}\textsc{abort} : U} \qquad \frac{\Psi;\Delta \vdash N_1 : A^+\ ord}{\Psi;\Delta \vdash [\![N_1]\!]^{A^+}\eta_{A^+}(z.\textsc{inl}(z)) : A^+ \oplus B^+\ lvl}$$

$$\frac{\Psi;\Theta\{x_1{:}{\uparrow}A^+\ ord\} \vdash N_1 : U \quad \Psi;\Theta\{x_1{:}{\uparrow}B^+\ ord\} \vdash N_2 : U}{\Psi;\Theta\{\!\{x{:}{\uparrow}(A^+ \oplus B^+)\}\!\} \vdash x \cdot {\uparrow}[\![[\eta_{A^+}(y.{\downarrow}{\uparrow}y), \eta_{B^+}(y.{\downarrow}{\uparrow}y)]]\!][{\downarrow}x_1.N_1, {\downarrow}x_2.N_2] : U}$$

$$\frac{}{\Psi;\Delta \vdash {\downarrow}\top : {\downarrow}\top\ lvl} \qquad \frac{\Psi;\Theta\{x_1{:}A^-\ ord\} \vdash N_1 : U}{\Psi;\Theta\{\!\{x{:}A^- \mathbin{\&} B^-\}\!\} \vdash [\![\eta_{A^-}(x \cdot \pi_1; \textsc{nil})/x_1]\!]N_1 : U}$$

$$\frac{\Psi;\Delta \vdash N_1 : {\downarrow}A^-\ ord \quad \Psi;\Delta \vdash N_2 : {\downarrow}B^-\ ord}{\Psi;\Delta \vdash [\![({\uparrow}N_1 \mathbin{\&} {\uparrow}N_2)/x]\!]{\downarrow}(\eta_{A^-}(x \cdot \pi_1; {\uparrow}({\downarrow}y.\, y \cdot \textsc{nil})) \mathbin{\&} \eta_{B^-}(\ldots)) : {\downarrow}(A^- \mathbin{\&} B^-)\ lvl}$$

Figure 3.10: Unfocused admissibility for the additive connectives of OL_3 (omits $\oplus_{R2}$, $\&_{L2}$)

$$\frac{\Psi \vdash t : \tau \quad \Psi;\Delta \vdash N : [t/a]A^+\ ord}{\Psi;\Delta \vdash [\![N]\!](\eta_{[t/a]A^+}(z.\,(t, z))) : \exists a{:}\tau.A^+\ lvl}$$

$$\frac{\Psi, a{:}\tau;\Theta\{x'{:}{\uparrow}A^+\ ord\} \vdash N : U}{\Psi;\Theta\{\!\{x{:}{\uparrow}(\exists a{:}\tau.A^+)\}\!\} \vdash x \cdot {\uparrow}[\![a.\eta_{A^+}(z.\,(a, {\downarrow}{\uparrow}z))]\!]^{\exists a:\tau.{\downarrow}{\uparrow}A^+}(a.{\downarrow}x'.N) : U}$$

$$\frac{\Psi, a{:}\tau;\Delta \vdash N : {\downarrow}A^-\ ord}{\Psi;\Delta \vdash [\![[a].{\uparrow}N/x]\!]^{\forall a:\tau.{\uparrow}{\downarrow}A^-}{\uparrow}([a].\eta_{A^-}(x \cdot [a]; {\uparrow}({\downarrow}y.y \cdot \textsc{nil}))) : {\downarrow}(\forall a{:}\tau.A^-)\ lvl}$$

$$\frac{\Psi \vdash t : \tau \quad \Psi;\Theta\{x'{:}[t/a]A^-\ ord\} \vdash N : U}{\Psi;\Theta\{\!\{x{:}\forall a{:}\tau.A^-\}\!\} \vdash [\![\eta_{[t/a]A^-}(x \cdot [a]; \textsc{nil})/x']\!]N' : U} \qquad \frac{}{\Psi;\cdot \vdash \textsc{refl} : t \doteq t\ lvl}$$

$$\frac{\forall(\Psi' \vdash \sigma : \Psi).\quad \sigma t = \sigma s \quad \to \quad \Psi';\sigma\Theta\{\cdot\} \vdash \phi(\sigma) : \sigma U}{\Psi;\Theta\{\!\{x{:}{\uparrow}(t \doteq s)\}\!\} \vdash x \cdot {\uparrow}(\textsc{unif}\,(\mathsf{fn}\ \sigma \Rightarrow \phi(\sigma))) : U}$$

Figure 3.11: Unfocused admissibility for the first-order connectives of OL_3

presentation, we present unfocused admissibility primarily on the level of proof terms. The resulting presentation is quite dense; proofs of this variety really ought to be mechanized, though we leave that for future work.

For the most part, there is exactly one unfocused admissibility rule for each rule of unfocused OL_3. The justifications for the unfocused admissibility lemmas for the multiplicative, exponential fragment of OL_3 are given in Figure 3.9; the additive fragment is given in Figure 3.10, and the first-order connectives are treated in Figure 3.11. There are two additional rules that account for the fact that different polarized propositions, like ${\downarrow}{\uparrow}{\downarrow}{\uparrow}A^+$ and A^+ erase to the same unpolarized proposition $(A^+)^\circ$. For the same reason, Figure 3.9 contains four id-like rules, since atomic propositions can come in positive and negative varieties and can appear in the context either suspended or not.

We can view unfocused admissibility as creating an abstraction layer of admissible rules that can be used to build focused proofs of stable sequents. The proof of the completeness of focusing below constructs focused proofs entirely by working through the interface layer of unfocused admissibility.

3.6.4 Focalization

The act of taking an unfocused proof of an erased sequent and getting a focused proof of the unerased sequent is *focalization*. If we run the constructive content of the proof of the completeness of focusing (the OL_3 analogue of Theorem 2.6 from Section 2.3.7), which takes any stable, suspension-normal sequent as input, the proof performs focalization.

Theorem 3.9 (Compltleness of focusing/focalization)**.**
If $\Psi; \Delta^\circ \Longrightarrow U^\circ$*, where* Δ *and* U *are stable and suspension-normal, then* $\Psi; \Delta \vdash U$*.*

Proof. By an outer induction on the structure of unfocused proofs and an inner induction over the structure of polarized formulas A^+ and A^- in order to remove series of shifts $\uparrow\downarrow\ldots\uparrow\downarrow A^-$ from formulas until an unfocused admissibility lemma can be applied. $\square$

3.7 Properties of syntactic fragments

In the structural focalization methodology, once cut admissibility and identity expansion are established the only interesting part of the proof of the completeness of focusing is the definition of an erasure function and the presentation of a series of unfocused admissibility lemmas. The unfocused admissibility lemmas for non-invertible rules, like $\bullet_R$ and $\rightarrowtail_L$, look straightforward:

$$\frac{\Psi; \Delta_1 \vdash A^+\ \mathit{true} \quad \Psi; \Delta_2 \vdash B^+\ \mathit{true}}{\Psi; \Delta_1, \Delta_2 \vdash A^+ \bullet B^+\ \mathit{lvl}} \qquad \frac{\Psi; \Delta_A \vdash A^+\ \mathit{ord} \quad \Psi; \Theta\{x'{:}B^-\ \mathit{ord}\} \vdash U}{\Psi; \Theta\{\!\{\Delta_A, x{:}A^+ \rightarrowtail B^-\}\!\} \vdash U}$$

Because unfocused admissibility is defined only on stable sequents in our methodology, the invertible rules, like $\bullet_L$ and $\rightarrowtail_R$, require the presence of shifts:

$$\frac{\Psi; \Theta\{x_1{:}\uparrow A^+\ \mathit{ord}, x_2{:}\uparrow B^+\ \mathit{ord}\} \vdash U}{\Psi; \Theta\{\!\{x{:}\uparrow(A^+ \bullet B^+)\}\!\} \vdash U} \qquad \frac{\Psi; x{:}\uparrow A^+\ \mathit{ord}, \Delta \vdash \downarrow B^-\ \mathit{true}}{\Psi; \Delta \vdash \downarrow(A^+ \rightarrowtail B^-)\ \mathit{lvl}}$$

The presence of shifts is curious, due to our observation in Section 3.3.2 that the shifts have much of the character of exponentials; they are exponentials that do not place any restrictions on the form of the context.

As a thought experiment, imagine the removal of shifts $\uparrow$ and $\downarrow$ from the language of propositions in OL_3. Were it not for the presence of atomic propositions p^+ and p^-, this change would make every proposition A^+ a mobile proposition A^+_{eph} and would make every proposition A^- a right-permeable proposition A^-_{lax}. But arbitrary atomic propositions are intended to be stand-ins for arbitrary propositions! If arbitrary propositions lack shifts, then non-mobile atomic propositions would appear to no longer stand for anything. Therefore, let's remove them too, leaving only the permeable, mobile, and right-permeable atomic propositions p^+_{pers}, p^+_{eph}, and p^-_{lax}. Having done so, every positive proposition is mobile, and every negative proposition is right-permeable.

Now we have a logical fragment where every positive proposition is mobile and every negative proposition is observed to be right-permeable. Consider a derivation $\Psi; \Delta \vdash A^+\ \mathit{lax}$ where Δ is stable and includes only linear and persistent judgments (that is, $\Delta\lceil_{\mathit{eph}}$). It is simple to observe that, for every subderivation $\Psi'; \Delta' \vdash U'$, if Δ' is stable then $\Delta' = \Delta'\lceil_{eph}$, and if U is stable then $U = U\lfloor^{lax}$. Given that this is the case, the restrictions that the focused $¡_R$ and $\bigcirc_L$ rules make are *always satisfiable*, the same property that we previously observed of focused shift rules $\downarrow_R$ and $\uparrow_L$. In our syntactic fragment, in other words, the exponentials ¡ and $\bigcirc$ have become effective replacements for $\downarrow$ and $\uparrow$.

The cut and identity theorems survive our restriction of the logic entirely intact: these theorems handle each of the connectives separately and are stable to the addition or removal of individual connectives. That is not true for the unfocused admissibility lemmas, which critically and heavily use shifts. However, while we no longer have our original shifts, we have replacement shifts in the form of ¡ and $\bigcirc$, and can replay the logic of the unfocused admissibility lemmas in order to gain new ones that look like this:

$$\frac{\Psi; \Delta_1 \vdash A^+ \, lax \quad \Psi; \Delta_2 \vdash B^+ \, lax}{\Psi; \Delta_1, \Delta_2 \vdash A^+ \bullet B^+ \, lax} \qquad \frac{\Psi; \Delta_A \vdash A^+ \, lax \quad \Psi; \Theta\{x'{:}B^- \, eph\} \vdash U}{\Psi; \Theta\{\!\{\Delta_A, x{:}A^+ \rightarrowtail B^-\}\!\} \vdash U}$$

$$\frac{\Psi; \Theta\{x_1{:}¡A^+ \, eph, x_2{:}¡B^+ \, eph\} \vdash U}{\Psi; \Theta\{\!\{x{:}¡(A^+ \bullet B^+)\}\!\} \vdash U} \qquad \frac{\Psi; x{:}¡A^+ \, eph, \Delta \vdash \bigcirc B^- \, lax}{\Psi; \Delta \vdash \bigcirc(A^+ \rightarrowtail B^-) \, lax}$$

(To be clear, just as all the unfocused admissibility lemmas only applied to stable sequents, the unfocused admissibility lemmas above only apply when contexts and succedents are both stable and free of judgments $T \, ord$ and $T \, true$.)

The point of this exercise is that, given the definition and metatheory of OL_3, there is a reasonably large family of related systems, including ordered linear logic, lax logic, linear lax logic, and linear logic, that can be given erasure-based focalization proofs relative to OL_3; at most, the erasure function and the unfocused admissibility lemmas need to be adapted. The fragment we have defined here corresponds to regular linear logic. In the erasure of polarized OL_3 propositions to linear logic propositions, the "pseudo-shifts" $\bigcirc$ and ¡ are wiped away: $(\bigcirc A^+)^\circ = (A^+)^\circ$ and $(¡A^-)^\circ = (A^-)^\circ$. Additionally, the two implications are conflated: $(A^+ \rightarrowtail B^-)^\circ = (A^+ \twoheadrightarrow B^-)^\circ = (A^+)^\circ \multimap (B^-)^\circ$. Beyond that, and the renaming of fuse to tensor – $(A^+ \bullet B^+)^\circ = (A^+)^\circ \otimes (B^+)^\circ$ – the structure of erasure remains intact, and we can meaningfully focalize unfocused linear logic derivations into focused OL_3 derivations.

3.8 The design space of proof terms

In the design space of logical frameworks, our decision to view proof terms E as being fully intrinsically typed representatives of focused derivations is somewhat unusual. This is because, in a dependently typed logical framework, the variable substitution theorem (which we had to establish very early on) and the cut admissibility theorem (which we established much later) are effectively the same theorem; handling everything at once is difficult at best, and dependent types seem to force everything to be handled at once in an intrinsically typed presentation.

Since the advent of Watkins' observations about the existence of hereditary substitution and its application to logical frameworks [WCPW02], the dominant approach to the metatheory of logical frameworks has to define proof terms E that have little, if any, implicit type structure: just enough so that it is possible to define the hereditary substitution function $[\![M/x]\!]E$. The work by Martens and Crary goes further, treating hereditary substitution as a relation, not a function, so that absolutely no intrinsic type system is necessary, and the proof terms are merely untyped abstract binding trees [MC12].

If we were to take such an approach, we would need to treat the judgment $\Psi;\Delta \vdash E : U$ as a genuine four-place relation, rather than the three-place relation $\Psi;\Delta \vdash U$ annotated with a derivation E of that sequent. Then, the analogue of cut admissibility (part 4) would show that if $\Psi;\Delta \vdash M : A^-$ and $\Psi;\Theta\{x{:}A^-\ lvl\} \vdash E : U$, and Ξ matches $\Theta\{\!\{\Delta\restriction_{lvl}\}\!\}$, then $\Psi;\Xi \vdash [\![M/x]\!]E : U$, where $[\![M/x]\!]E$ *is some function on proof terms that has already been defined*, rather than just an expression of the computational content of the theorem. Being able to comfortably conflate the computational content of a theorem with its operation on proof terms is the primary advantage of the approach taken in this chapter; it avoids a great deal of duplicated effort. The cost to this approach is that we cannot apply the modern Canonical LF methodology in which we define a proof term language that is intrinsically only simply well-typed and then overlay a dependent type system on top of it (this is discussed in Section 4.1.2 in the context of LF). As we discuss further in Section 4.7.3, this turns out not to be a severe limitation given the way we want to use OL_3.

It is not immediately obvious how the substitution $[\![M/x]\!]E$ could be defined without accounting for the full structure of derivations. The rightist substitution function, in particular, is computationally dependent on the implicit bookkeeping associated with the matching constructs, and that bookkeeping is more of an obstacle in our setting than the implicit type annotations. The problem, if we wish to see it as a problem, is that we cannot substitute a derivation M of $\Psi;\Delta \vdash A^-$ into a derivation E of $\Psi;\Theta\{x{:}A^-\ ord\} \vdash U$ unless x is *actually free* in E. Therefore, when we try to substitute the same M into $V_1 \bullet V_2$, we are forced to determine what judgment x is associated with; if x is associated with a linear or ephemeral judgment, we must track which subderivation x is assigned to in order to determine what is to be done next.

Fine-grained tracking of variables during substitution is both very inefficient when type theories are implemented as logical frameworks and unnatural to represent for proof assistants like Twelf that implement a persistent notion of bound variables. Therefore other developments have addressed this problem (see, for example, Cervesato et al. [CdPR99], Schack-Nielsen and Schürmann [SNS10], and Crary [Cra10]). It might be possible to bring our development more in line with these other developments by introducing a new matching construct of substitution into contexts, the substitution construct $[\Delta/(x{:}A^-\ lvl)]\Xi$. If $\Xi = \Theta\{x{:}A^-\ lvl\}$, then this would be the same as $\Theta\{\!\{\Delta\}\!\}$, but if x is not in the variable domain of Ξ, then Ξ matches $[\Delta/(x{:}A^-\ lvl)]\Xi$.

$$\frac{\Delta\restriction_{lvl} \quad \Psi;\Delta \vdash M : A^- \quad \Psi;\Xi \vdash E : U \quad \Delta\ stable_L \quad [\![M/x]\!]E = E'}{\Psi;[\Delta/(x{:}A^-\ lvl)]\Xi \vdash E' : U}\ rcut$$

Using this formulation of *rcut*, it becomes unproblematic to define $[\![M/x]\!](V_1 \bullet V_2)$ as substituting M into both V_1 and V_2, as we are allowed to substitute for x even in terms where the variable cannot appear. Using this strategy, it should be possible to describe and formalize the development in this chapter with proof terms that do nothing more than capture the binding structure of derivations.

The above argument suggests that the framing-off operation is *inconvenient* to use for specifying the *rcut* part of cut admissibility, because it forces us to track where the variable ends up and direct the computational content of cut admissibility accordingly. However, the development in this chapter shows that it is clearly possible to define cut admissibility in terms of the framing-off operation $\Theta\{\!\{\Delta\}\!\}$. That is not necessarily the case for every logic. For instance, to give a focused presentation of Reed's queue logic [Ree09], we would need a matching construct $[\Delta/x]\Xi$ that is quite different from the framing-off operation $\Delta\{\!\{x{:}A^-\}\!\}$ used to describe the logic's left rules. I conjecture that logics where the framing-off operation is adequate for the presentation of cut admissibility are the same as those logics which can be treated in Belnap's display logic [Bel82].

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