Chapter 2

Linear logic

In this chapter, we present linear logic as a logic with the ability to express aspects of state and state transition in a natural way. In Chapter 3 we will repeat the development from this chapter in a much richer and more expressive setting, and in Chapter 4 we will carve out a fragment of this logic to use as the basis of SLS, our logical framework of substructural logical specifications. These three chapters contribute to the overall thesis by focusing on the design of logical frameworks:

> **Thesis (Part I):** *The methodology of structural focalization facilitates the derivation of logical frameworks as fragments of focused logics.*

The purpose of this chapter is to introduce the methodology of *structural focalization*; this development is one of the major contributions of this work. Linear logic is a fairly simple logic that nevertheless allows us to consider many of the issues that will arise in richer substructural logics like the one considered in Chapter 3.

In Section 2.1 we motivate and discuss a traditional account of linear logic, and in Section 2.2 we discuss why this account is insufficient as a *logical framework* – derivations in linear logic suffice to establish the existence of a series of state transitions but do not adequately capture the structure of those transitions. Our remedy for this insufficiency comes in the form of *focusing*, Andreoli's restricted normal form for derivations in linear logic. We discuss focusing for a polarized presentation of linear logic in Section 2.3.

With focusing, we can describe *synthetic inference rules* (Section 2.4) that succinctly capture the structure of focused transitions. In Section 2.5 we discuss a number of ways of modifying the design of our focused logic to increase the expressiveness of synthetic inference rules; one of the alternatives we present, the introduction of *permeable atomic propositions*, will be generalized and incorporated into the focused presentation of ordered linear lax logic that we discuss in Chapter 3.

2.1 Introduction to linear logic

Logic as it has been traditionally understood and studied – both in its classical and intuitionistic varieties – treats the truth of a proposition as a *persistent resource*. That is, if we have evidence

$$\begin{array}{lll} A & ::= p \mid\ !A \mid \mathbf{1} \mid A \otimes B \mid A \multimap B & \\ \Gamma & ::= \cdot \mid \Gamma, A & \textit{(multiset)} \\ \Delta & ::= \cdot \mid \Delta, A & \textit{(multiset)} \end{array}$$

$\boxed{\Gamma; \Delta \longrightarrow A}$

$$\frac{}{\Gamma; p \longrightarrow p}\ id \qquad \frac{\Gamma, A; \Delta, A \longrightarrow C}{\Gamma, A; \Delta \longrightarrow C}\ copy$$

$$\frac{\Gamma; \cdot \longrightarrow A}{\Gamma; \cdot \longrightarrow !A}\ !_R \qquad \frac{\Gamma, A; \Delta \longrightarrow C}{\Gamma; \Delta, !A \longrightarrow C}\ !_L \qquad \frac{}{\Gamma; \cdot \longrightarrow \mathbf{1}}\ \mathbf{1}_R \qquad \frac{\Gamma; \Delta \longrightarrow C}{\Gamma; \Delta, \mathbf{1} \longrightarrow C}\ \mathbf{1}_L$$

$$\frac{\Gamma; \Delta_1 \longrightarrow A \quad \Gamma; \Delta_2 \longrightarrow B}{\Gamma; \Delta_1, \Delta_2 \longrightarrow A \otimes B}\ \otimes_R \qquad \frac{\Gamma; \Delta, A, B \longrightarrow C}{\Gamma; \Delta, A \otimes B \longrightarrow C}\ \otimes_L$$

$$\frac{\Gamma; \Delta, A \longrightarrow B}{\Gamma; \Delta \longrightarrow A \multimap B}\ \multimap_R \qquad \frac{\Gamma; \Delta_1 \longrightarrow A \quad \Gamma; \Delta_2, B \longrightarrow C}{\Gamma; \Delta_1, \Delta_2, A \multimap B \longrightarrow C}\ \multimap_L$$

Figure 2.1: Intuitionstic linear logic

for the truth of a proposition, we can ignore that evidence if it is not needed and reuse the evidence as many times as we need to. Throughout this document, "logic as it has been traditionally understood as studied" will be referred to as *persistent* logic to emphasize this treatment of evidence.

Linear logic, which was studied and popularized by Girard [Gir87], treats evidence as an *ephemeral* resource; the use of an ephemeral resource consumes it, at which point it is unavailable for further use. Linear logic, like persistent logic, comes in classical and intuitionistic flavors. We will favor intuitionistic linear logic in part because the propositions of intuitionistic linear logic (written A, B, C, $\ldots$) have a more natural correspondence with our physical intuitions about consumable resources. Linear conjunction $A \otimes B$ ("A tensor B") represents the resource built from the resources A and B; if you have both a bowl of soup *and* a sandwich, that resource can be represented by the proposition $\mathsf{soup} \otimes \mathsf{sandwich}$. Linear implication $A \multimap B$ ("A lolli B") represents a resource that can interact with another resource A to produce a resource B. One robot with batteries not included could be represented as the linear resource $(\mathsf{battery} \multimap \mathsf{robot})$, and the linear resource $(\mathsf{6bucks} \multimap \mathsf{soup} \otimes \mathsf{sandwich})$ represents the ability to use \$6 to obtain lunch – but only once.[1] Linear logic also has a connective $!A$ ("bang A" or "of course A") representing a persistent resource that can be used to generate any number of A resources, including zero. Your local Panera, which allows six dollars to be exchanged for both soup and a sandwich any number of times, can be represented as the resource $!(\mathsf{6bucks} \multimap \mathsf{soup} \otimes \mathsf{sandwich})$.

Figure 2.1 presents a standard sequent calculus for linear logic, in particular the *multiplica-*

[1] Conjunction will always bind more tightly than implication, so this is equivalent to the proposition $\mathsf{6bucks} \multimap (\mathsf{soup} \otimes \mathsf{sandwich})$.

tive, exponential fragment of intuitionistic linear logic (or *MELL*), so called because the connectives $\mathbf{1}$, $A \otimes B$, and $A \multimap B$ are considered to be the *multiplicative* connectives, and the connective $!A$ is the *exponential* connective of intuitionistic linear logic.[2] It corresponds most closely to Barber's dual intuitionistic linear logic [Bar96], but also to Andreoli's dyadic system [And92] and Chang et al.'s judgmental analysis of intuitionistic linear logic [CCP03].

The propositions of intuitionistic linear logic, and linear implication in particular, capture a notion of state change: we can *transition* from a state where we have both a battery and the battery-less robot (represented, as before, by the linear implication $\mathsf{battery} \multimap \mathsf{robot}$) to a state where we have the battery-endowed (and therefore presumably functional) robot (represented by the proposition robot). In other words, the proposition

$$\mathsf{battery} \otimes (\mathsf{battery} \multimap \mathsf{robot}) \multimap \mathsf{robot}$$

is provable in linear logic. These transitions can be chained together as well: if we start out with $\mathsf{6bucks}$ instead of $\mathsf{battery}$ but we also have the persistent ability to turn $\mathsf{6bucks}$ into a battery – just like we turned \$6 into a bowl of soup and a salad at Panera – then we can ultimately get our working robot as well. Written as a series of transitions, the picture looks like this:

\$6 (1)		*battery (1)*		*robot (1)*
battery-less robot (1)	$\leadsto$	*battery-less robot (1)*	$\leadsto$	*turn \$6 into a battery (all you want)*
turn \$6 into a battery (all you want)		*turn \$6 into a battery (all you want)*		

In linear logic, these transitions correspond to the provability of the proposition

$$!(\mathsf{6bucks} \multimap \mathsf{battery}) \otimes \mathsf{6bucks} \otimes (\mathsf{battery} \multimap \mathsf{robot}) \multimap \mathsf{robot}.$$

A derivation of this proposition is given in Figure 2.2.[3]

It is precisely because linear logic contains this intuitive notion of state and state transition that a rich line of work, dating back to Chirimar's 1995 dissertation, has sought to use linear logic as a *logical framework* for describing stateful systems [Chi95, CP02, CPWW02, Pfe04, Mil09, PS09, CS09].

2.2 Logical frameworks

Generally speaking, logical frameworks use the *structure* of proofs in a logic (like linear logic) to describe the structures we're interested in (like the process of obtaining a robot). There are

[2] In this chapter we will mostly ignore the *additive* connectives of intuitionistic linear logic $\mathbf{0}$, $A \oplus B$, $\top$, and $A \mathbin{\&} B$ and will entirely ignore the *first-order* connectives $\exists x.A$ and $\forall x.A$. The "why not" connective $?A$ from classical linear logic is sometimes treated as a second exponential connective in intuitionistic linear logic [CCP03], but we will never ask "why not?" in the context of this dissertation.

[3] In Chapter 4 (and Section 4.7.2 in particular) we see that this view isn't quite precise enough, and that the "best" representation of state change from the state A to the state B isn't really captured by derivations of the proposition $A \multimap B$ or by derivations of the sequent $\cdot; A \longrightarrow B$. However, this view remains a simple and useful one; Cervesato and Scedrov cover it thoroughly in the context of intuitionistic linear logic [CS09].

$$
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  \dfrac{
   \dfrac{
    \dfrac{
     \dfrac{
      \dfrac{
       \dfrac{
        \dfrac{}{\Gamma; \mathsf{6bucks} \longrightarrow \mathsf{6bucks}}\,id
        \quad
        \dfrac{}{\Gamma; \mathsf{battery} \longrightarrow \mathsf{battery}}\,id
       }{\Gamma; \mathsf{6bucks}, \mathsf{6bucks} \multimap \mathsf{battery} \longrightarrow \mathsf{battery}}\,{\multimap_L}
      }{\Gamma; \mathsf{6bucks} \longrightarrow \mathsf{battery}}\,copy
      \quad
      \dfrac{}{\Gamma; \mathsf{robot} \longrightarrow \mathsf{robot}}\,id
     }{\Gamma; \mathsf{6bucks}, \mathsf{battery} \multimap \mathsf{robot} \longrightarrow \mathsf{robot}}\,{\multimap_L}
    }{\Gamma; \mathsf{6bucks} \otimes (\mathsf{battery} \multimap \mathsf{robot}) \longrightarrow \mathsf{robot}}\,{\otimes_L}
   }{\cdot; !(\mathsf{6bucks} \multimap \mathsf{battery}), \mathsf{6bucks} \otimes (\mathsf{battery} \multimap \mathsf{robot}) \longrightarrow \mathsf{robot}}\,{!_L}
  }{\cdot; !(\mathsf{6bucks} \multimap \mathsf{battery}) \otimes \mathsf{6bucks} \otimes (\mathsf{battery} \multimap \mathsf{robot}) \longrightarrow \mathsf{robot}}\,{\otimes_L}
 }{\cdot; \cdot \longrightarrow !(\mathsf{6bucks} \multimap \mathsf{battery}) \otimes \mathsf{6bucks} \otimes (\mathsf{battery} \multimap \mathsf{robot}) \multimap \mathsf{robot}}\,{\multimap_R}
}{}
$$

Figure 2.2: Proving that a transition is possible (where we let $\Gamma = \mathsf{6bucks} \multimap \mathsf{battery}$)

two related reasons why linear logic as described in Figure 2.1 is not immediately useful as a logical framework. First, the structure of the derivation in Figure 2.2 doesn't really match the intuitive two-step transition that we sketched out above. Second, there are *lots* of derivations of our example proposition according to the rules in Figure 2.1, even though there's only one "real" series of transitions that get us to a working robot. The use of $!L$, for instance, could be permuted up past the $\otimes L$ and then past the $\multimap L$ into the left branch of the proof. These differences represent inessential nondeterminism in proof construction – they just get in the way of the structure that we are trying to capture.

This is a general problem in the construction of logical frameworks. We'll discuss two solutions in the context of LF, a logical framework based on dependent type theory that has proved to be a suitable means of encoding a wide variety of deductive systems, such as logics and programming languages [HHP93]. The first solution is to define an appropriate equivalence class of proofs, and the second solution is to define a complete set of canonical proofs.

Defining an appropriate equivalence relation on proofs can be an effective way of handling this inessential nondeterminism. In linear logic as presented above, if the permutability of rules like $!_L$ and $\otimes_L$ is problematic, we can instead reason about *equivalence classes* of derivations. Derivations that differ only in the ordering of $!_L$ and $\otimes_L$ rules belong in the same equivalence class (which means we treat them as equivalent):

$$
\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma, A; \Delta, B, C \longrightarrow D\end{matrix}}{\Gamma, A; \Delta, B \otimes C \longrightarrow D}\,{\otimes_L}}{\Gamma; \Delta, !A, B \otimes C \longrightarrow D}\,{!_L}
\quad\equiv\quad
\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma, A; \Delta, B, C \longrightarrow D\end{matrix}}{\Gamma; \Delta, !A, B, C \longrightarrow D}\,{!_L}}{\Gamma; \Delta, !A, B \otimes C \longrightarrow D}\,{\otimes_L}
$$

In LF, lambda calculus terms (which correspond to derivations by the Curry-Howard correspondence) are considered modulo the least equivalence class that includes

* α-equivalence ($\lambda x.N \equiv \lambda y.N[y/x]$ if $y \notin FV(N)$),
* β-equivalence ($(\lambda x.\, M)N \equiv M[N/x]$ if $x \notin FV(N)$), and
* η-equivalence ($N \equiv \lambda x.N\,x$).

The weak normalization property for LF establishes that, given any typed LF term, we can find an equivalent term that is β-normal (no β-redexes of the form $(\lambda x.M)N$ exist) and η-long (replacing N with $\lambda x.N\,x$ anywhere would introduce a β-redex or make the term ill-typed). In any given equivalence class of typed LF terms, all the β-normal and η-long terms are α-equivalent. Therefore, because α-equivalence is decidable, the equivalence of typed LF terms is decidable.

The uniqueness of β-normal and η-long terms within an equivalence class of lambda calculus terms (modulo α-equivalence, which we will henceforth take for granted) makes these terms useful as canonical representatives of equivalence classes. In Harper, Honsell, and Plotkin's original formulation of LF, a deductive system is said to be *adequately encoded* as an LF type family in the case that there is a compositional bijection between the formal objects in the deductive system and these β-normal, η-long representatives of equivalence classes [HHP93]. (Adequacy is a topic we will return to in Section 4.1.4.)

Modern presentations of LF, such as Harper and Licata's [HL07], follow the approach developed by Watkins et al. [WCPW02] and define the logical framework so that it only contains these β-normal, η-long *canonical forms* of LF. This presentation of LF is called Canonical LF to distinguish it from the original presentation of LF in which the β-normal, η-long terms are just a refinement of terms. A central component in this approach is *hereditary substitution*; in Chapter 3, we will make the connections between hereditary substitution and the focused cut admissibility property we prove in this chapter more explicit. Hereditary substitution also establishes a normalization property for LF. Using hereditary substitution we can easily take a regular LF term and transform it into a Canonical LF term. By a separate theorem, we can prove that the normalized term will be equivalent to the original term [MC12].

Our analogue to the canonical forms of LF will be the *focused derivations* of linear logic that are presented in the next section. In Section 2.3 below, we present focused linear logic and see that there is exactly one focused derivation of the proposition

$$!(\mathsf{6bucks} \multimap \mathsf{battery}) \otimes \mathsf{6bucks} \otimes (\mathsf{battery} \multimap \mathsf{robot}) \multimap \mathsf{robot}.$$

We will furthermore see that the structure of this derivation matches the intuitive transition interpretation, a point that is reinforced by the discussion of *synthetic inference rules* in Section 2.4.

2.3 Focused linear logic

Andreoli's original motivation for introducing focusing was not to describe a logical framework, it was to describe a foundational logic programming paradigm based on proof search in classical linear logic [And92]. The existence of multiple proofs that differ in inessential ways is particularly problematic for proof search, as inessential differences between derivations correspond to unnecessary choice points that a proof search procedure will need to backtrack over.

The development in this section introduces *structural focalization*, a methodology for deriving the correctness of a focused sequent calculus (Theorem 2.5 and Theorem 2.6, Section 2.3.7) as a consequence of the internal completeness (identity expansion, Theorem 2.3, Section 2.3.5) and internal soundness (cut admissibility, Theorem 2.4, Section 2.3.6) of the focused system. This methodology is a substantial refinement of the method used by Chaudhuri to establish the correctness of focused intuitionistic linear logic [Cha06], and because it relies on structural methods, structural focalization is more amenable to mechanized proof [Sim11]. Our focused sequent calculus also departs from Chaudhuri's by treating asynchronous rules as confluent rather than fixed, a point that will be discussed in Section 2.3.8.

2.3.1 Polarity

The first step in describing a focused sequent calculus is to classify connectives into two groups [And92]. Some connectives, such as linear implication $A \multimap B$, are called *asynchronous* because their right rules can always be applied eagerly, without backtracking, during bottom-up proof search. Other connectives, such as multiplicative conjunction $A \otimes B$, are called *synchronous* because their right rules cannot be applied eagerly. For instance, if we are trying to prove the sequent $A \otimes B \longrightarrow B \otimes A$, the $\otimes R$ rule cannot be applied eagerly; we first have to decompose $A \otimes B$ on the left using the $\otimes L$ rule. The terms asynchronous and synchronous make a bit more sense in a one-sided classical sequent calculus; in intuitionistic logics, it is common to call asynchronous connectives *right*-asynchronous and *left*-synchronous. Similarly, it is common to call synchronous connectives *right*-synchronous and *left*-asynchronous. We will instead use a different designation, calling the (right-)synchronous connectives *positive* ($!$, $\mathbf{0}$, $\oplus$, $\mathbf{1}$, and $\otimes$ in full propositional linear logic) and calling the (right-)asynchronous connectives *negative* ($\multimap$, $\top$ and $\&$ in full propositional linear logic); this assignment is called the proposition's *polarity*. Each atomic proposition must be assigned to have only one polarity, though this assignment can be made arbitrarily.

The nontrivial result of focusing is that it is possible to separate a proof into two strictly alternating phases. In *inversion* phases, positive propositions on the left and negative propositions on the right are eagerly and exhaustively decomposed using invertible rules.[4] In *focused* phases, a single proposition is selected (the proposition *in focus*, which is either a positive proposition in right focus or a negative proposition in left focus). This proposition is then decomposed repeatedly and exhaustively using rules that are mostly non-invertible.

If we consider this discipline applied to our robot example where all atoms have been assigned positive polarity, we would begin with an inversion phase, decomposing the negative implication on the right and the positive tensor and exponential on the left:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\mathsf{6bucks} \multimap \mathsf{battery}; \mathsf{6bucks}, \mathsf{battery} \multimap \mathsf{robot} \longrightarrow \mathsf{robot}}}{\mathsf{6bucks} \multimap \mathsf{battery}; \mathsf{6bucks} \otimes (\mathsf{battery} \multimap \mathsf{robot}) \longrightarrow \mathsf{robot}}\,\otimes_L}{\cdot; !(\mathsf{6bucks} \multimap \mathsf{battery}), \mathsf{6bucks} \otimes (\mathsf{battery} \multimap \mathsf{robot}) \longrightarrow \mathsf{robot}}\,!_L}{\cdot; !(\mathsf{6bucks} \multimap \mathsf{battery}) \otimes \mathsf{6bucks} \otimes (\mathsf{battery} \multimap \mathsf{robot}) \longrightarrow \mathsf{robot}}\,\otimes_L
$$
$$
\dfrac{\cdot; !(\mathsf{6bucks} \multimap \mathsf{battery}) \otimes \mathsf{6bucks} \otimes (\mathsf{battery} \multimap \mathsf{robot}) \longrightarrow \mathsf{robot}}{\cdot; \cdot \longrightarrow !(\mathsf{6bucks} \multimap \mathsf{battery}) \otimes \mathsf{6bucks} \otimes (\mathsf{battery} \multimap \mathsf{robot}) \multimap \mathsf{robot}}\,\multimap_R
$$

Once we reach the topmost sequent in the above fragment, we have to pick a negative proposition on the left or a positive proposition on the right as our focus in order to proceed. The correct choice in this context is to pick the negative proposition $\mathsf{6bucks} \multimap \mathsf{battery}$ in the persistent context and decompose it using the non-invertible rule $\multimap_L$. Because the subformula $\mathsf{6bucks}$ is

[4] Synchronicity or polarity, a property of connectives, is closely connected to (and sometimes conflated with) a property of rules called *invertibility*; a rule is invertible if the conclusion of the rule implies the premises. So $\multimap R$ is invertible ($\Gamma; \Delta \longrightarrow A \multimap B$ implies $\Gamma; \Delta, A \longrightarrow B$) but $\multimap L$ is not ($\Gamma; \Delta, A \multimap B \longrightarrow C$ does not imply that $\Delta = \Delta_1, \Delta_2$ such that $\Gamma; \Delta_1 \longrightarrow A$ and $\Gamma; \Delta_2, B \longrightarrow C$). Rules that can be applied eagerly need to be invertible, so asynchronous connectives have invertible right rules and synchronous connectives have invertible left rules. Therefore, in the literature a common synonym for asynchronous/negative is *right-invertible*, and the analogous synonym for synchronous/positive is *left-invertible*.

$$\begin{array}{rcl|rcl|rcl}
(\downarrow A^-)^\circ & = & (A^-)^\circ & & & & & & \\
(p^+)^\circ & = & p^+ & (p^+)^\oplus & = & p^+ & (p^+)^\ominus & = & \uparrow p^+ \\
(!A^-)^\circ & = & !(A^-)^\circ & (!A)^\oplus & = & !A^\ominus & (!A)^\ominus & = & \uparrow(!A^\ominus) \\
(\mathbf{1})^\circ & = & \mathbf{1} & (\mathbf{1})^\oplus & = & \mathbf{1} & (\mathbf{1})^\ominus & = & \uparrow\mathbf{1} \\
(A^+ \otimes B^+)^\circ & = & (A^+)^\circ \otimes (B^+)^\circ & (A \otimes B)^\oplus & = & A^\oplus \otimes B^\oplus & (A \otimes B)^\ominus & = & \uparrow(A^\oplus \otimes B^\oplus) \\
(\uparrow A^+)^\circ & = & (A^+)^\circ & & & & & & \\
(p^-)^\circ & = & p^- & (p^-)^\oplus & = & \downarrow p^- & (p^-)^\ominus & = & p^- \\
(A^+ \multimap B^-)^\circ & = & (A^+)^\circ \multimap (B^-)^\circ & (A \multimap B)^\oplus & = & \downarrow(A^\oplus \multimap B^\ominus) & (A \multimap B)^\ominus & = & A^\oplus \multimap B^\ominus
\end{array}$$

Figure 2.3: De-polarizing and polarizing (with minimal shifts) propositions of MELL

positive and ends up on the right side in the subderivation, the focusing discipline requires that we prove it immediately with the id rule. Letting $\Gamma = \mathsf{6bucks} \multimap \mathsf{battery}$, this looks like this:

$$\dfrac{\dfrac{\dfrac{}{\Gamma; \mathsf{6bucks} \longrightarrow \mathsf{6bucks}}\,id \quad \dfrac{\vdots}{\Gamma; \mathsf{battery} \multimap \mathsf{robot}, \mathsf{battery} \longrightarrow \mathsf{robot}}}{\Gamma; \mathsf{6bucks}, \mathsf{battery} \multimap \mathsf{robot}, \mathsf{6bucks} \multimap \mathsf{battery} \longrightarrow \mathsf{robot}}\,{\multimap_L}}{\Gamma; \mathsf{6bucks}, \mathsf{battery} \multimap \mathsf{robot} \longrightarrow \mathsf{robot}}\,copy$$
$$\vdots$$

The trace (that is, the pair of a single bottom sequent and a set of unproved top sequents) of an inversion phase stacked on top of a focused phase is called a *synthetic inference rule* by Chaudhuri, a point we will return to in Section 2.4.

2.3.2 Polarization

At this point, there is an important choice to make. One way forward is to treat positive and negative propositions as syntactic refinements of the set of all propositions, and to develop a focused presentation for intuitionistic linear logic with the connectives and propositions that we have already considered, as Chaudhuri did in [Cha06]. The other way forward is to treat positive and negative propositions as distinct syntactic classes A^+ and A^- with explicit inclusions, called *shifts*, between them. This is called *polarized* linear logic. The positive proposition $\downarrow A^-$, pronounced "downshift A" or "down A," has a subterm that is a negative proposition; the negative proposition $\uparrow A^+$, pronounced "upshift A" or "up A," has a subterm that is a positive proposition.

$$A^+ ::= p^+ \mid \downarrow A^- \mid !A^- \mid \mathbf{1} \mid A^+ \otimes B^+$$
$$A^- ::= p^- \mid \uparrow A^+ \mid A^+ \multimap B^-$$

The relationship between unpolarized and polarized linear logic is given by two erasure functions $(A^+)^\circ$ and $(A^-)^\circ$ that wipe away all the shifts; this function is defined in Figure 2.3. In the other direction, every proposition in unpolarized linear logic has an polarized analogue with a minimal number of shifts, given by the functions $A^\oplus$ and $A^\ominus$ in Figure 2.3. Both of these functions are partial inverses of erasure, since $(A^\oplus)^\circ = (A^\ominus)^\circ = A$; we will generally refer to partial

$$
\begin{array}{rcl|rcl}
(p^+)^{m+} & = & {\downarrow}{\uparrow} p^+ & (p^+)^{m-} & = & {\uparrow} p^+ \\
(!A)^{m+} & = & {\downarrow}{\uparrow} !(A)^{m-} & (!A)^{m-} & = & {\uparrow} !(A)^{m-} \\
(\mathbf{1})^{m+} & = & {\downarrow}{\uparrow} \mathbf{1} & (\mathbf{1})^{m-} & = & {\uparrow} \mathbf{1} \\
(A \otimes B)^{m+} & = & {\downarrow}{\uparrow}((A)^{m+} \otimes (B)^{m+}) & (A \otimes B)^{m-} & = & {\uparrow}(A^{\oplus} \otimes B^{\oplus}) \\
 & & & & & \\
(p^-)^{m+} & = & {\downarrow} p^- & (p^-)^{m-} & = & {\uparrow}{\downarrow} p^- \\
(A \multimap B)^{m+} & = & {\downarrow}((A)^{m+} \multimap (B)^{m-}) & (A \multimap B)^{m-} & = & {\uparrow}{\downarrow}((A)^{m+} \multimap (B)^{m-})
\end{array}
$$

Figure 2.4: Fully-shifting polarization strategy for MELL

inverses of erasure as *polarization strategies*. The strategies $A^{\oplus}$ and $A^{\ominus}$ are minimal, avoiding shifts wherever possible, but there are many other possible strategies, such as the fully-shifting strategy that always adds either one or two shifts between every connective, which we can write as $(A)^{m+} = B^+$ and $(A)^{m-} = B^-$, defined in Figure 2.4.

Shifts turn out to have a profound impact on the structure of focused proofs, though erasure requires that they have no impact on *provability*. For instance, the proofs of A in Chaudhuri's focused presentation of linear logic are isomorphic to the proofs of $(A)^{\oplus}$ in the polarized logic discussed below,[5] whereas the proofs of $(A)^{m+}$ in polarized logic are isomorphic to the *unfocused* proofs of linear logic as described in Figure 2.1. Other polarization strategies correspond to different focused logics, as explored by Liang and Miller in [LM09], so the presentation of polarized linear logic below, like Liang and Miller's LJF, can be seen in two ways: as a focused logic in its own right, and as a framework for defining many focused logics (one per polarization strategy). As such, the strongest statement of the correctness of focusing is based on erasure: there is an unfocused derivation of $(A^+)^{\circ}$ or $(A^-)^{\circ}$ if and only if there is a focused derivation of A^+ or A^-. Most existing proofs of the completeness of focusing only verify a weaker property: that there is an unfocused derivation of A if and only if there is a focused derivation of $A^{\bullet}$, where $A^{\bullet}$ is some polarization strategy. The only exception seems to be Zeilberger's proof for classical persistent logic [Zei08].

In this dissertation, we will be interested only in the structure of focused proofs, which corresponds to using the polarization strategy given by $A^{\oplus}$ and $A^{\ominus}$. Therefore, following Chaudhuri, it would be possible to achieve our objectives without the use of polarization. Our choice is largely based on practical considerations: the use of polarized logic simplifies the proof of identity expansion in Section 2.3.5 and the proof of completeness in Section 2.3.7. That said, polarized logic is an independently significant and currently active area of research. For instance, the Curry-Howard interpretation of polarized persistent logic has been studied by Levy as Call-by-Push-Value [Lev04]. The erasable influence of the shifts on the structure (but not the existence) of proofs is also important in the context of theorem proving. For instance, a theorem prover for polarized logic can imitate focused proof search by using the $(A)^{\oplus}$ polarization strategy and unfocused proof search by using the $(A)^{m+}$ polarization strategy [MP09].

[5] This isomorphism holds for Chaudhuri's focused presentation of linear logic precisely because his treatment of atomic propositions differs from Andreoli's. This isomorphism does not hold relative to focused systems that follow Andreoli's design, a point we will return to in Section 2.5.

2.3.3 Focused sequent calculus

Usually, focused logics are described as having multiple sequent forms. For intuitionistic logics, there need to be at least three sequent forms:

* $\Gamma; \Delta \vdash [A^+]$ (the *right focus* sequent, where the proposition A^+ is in focus),
* $\Gamma; \Delta \vdash C$ (the *inversion* sequent), and
* $\Gamma; \Delta, [A^-] \vdash C$ (the *left focus* sequent, where the proposition A^- is in focus).

It is also possible to distinguish a fourth sequent form, the *stable* sequents, inversion sequents $\Gamma; \Delta \vdash C$ where no asynchronous inversion remains to be done. A sufficient condition for stability is that the context Δ contains only negative propositions A^- and the succedent C is a positive proposition A^+. However, this cannot be a *necessary* condition for stability due to the presence of atomic propositions. If the process of inversion reaches a positive atomic proposition p^+ on the left or a negative atomic proposition p^- on the right, the proposition can be decomposed no further. When we reach an atomic proposition, we are therefore forced to *suspend* decomposition, either placing a suspended positive atomic proposition $\langle p^+ \rangle$ in Δ or placing a suspended negative proposition $\langle p^- \rangle$ as the succedent. For technical reasons discussed below in Section 2.3.4, our sequent calculus can handle arbitrary suspended propositions, not just suspended atomic propositions, and suspended propositions are always treated as stable, so $\Gamma; A^-, B^-, C^- \vdash D^+$ and $\Gamma; \langle A^+ \rangle, B^-, \langle C^+ \rangle \vdash \langle D^- \rangle$ are both stable sequents.

Another reasonable presentation of linear logic, and the one we will adopt in this section, uses only one sequent form, $\Gamma; \underline{\Delta} \vdash \underline{U}$, that generalizes what is allowed to appear in the linear context $\underline{\Delta}$ or in the succedent $\underline{U}$. We will use this interpretation to understand the logic described in Figure 2.5. In addition to propositions A^+, A^- and positive suspended positive propositions $\langle A^+ \rangle$, the grammar of contexts $\underline{\Delta}$ allows them to contain left focuses $[A^-]$. Likewise, a succedent $\underline{U}$ can be a stable positive proposition A^+, a suspended negative proposition $\langle A^- \rangle$, a focused positive proposition $[A^+]$, or an inverting negative proposition A^-. We will henceforth write Δ and U to indicate the refinements of $\underline{\Delta}$ and $\underline{U}$ that do not contain any focus.

By adding a side condition to the three rules focus_R, focus_L, and copy that neither the context Δ nor the succedent U can contain an in-focus proposition $[A^+]$ or $[A^-]$, derivations can maintain the invariant that there is always at most one proposition in focus in any sequent, effectively restoring the situation in which there are three distinct judgments. Therefore, from this point on, we will only consider sequents $\Gamma; \underline{\Delta} \vdash \underline{U}$ with at most one focus. Pfenning, who developed this construction in [Pfe12], calls this invariant the *focusing constraint*. The focusing constraint alone gives us what Pfenning calls a *chaining* logic [Pfe12] and which Laurent calls a *weakly focused* logic [Lau04].[6] We obtain a fully focused logic by further restricting the three critical rules focus_R, focus_L, and copy so that they only apply when the sequent below the line is stable. In light of this additional restriction, whenever we consider a focused sequent $\Gamma; \Delta, [A^-] \vdash U$ or $\Gamma; \Delta \vdash [A^+]$, we can assume that Δ and U are stable.

[6] Unfortunately, I made the meaning of "weak focusing" less precise by calling a different sort of logic weakly focused in [SP11b]. That weakly focused system had an additional restriction that invertible rules could *not* be applied when any other proposition was in focus, which is what Laurent called a strongly $+$-focused logic.

$$
\begin{array}{lll}
A^+ & ::= p^+ \mid {\downarrow}A^- \mid\ !A^- \mid \mathbf{1} \mid A^+ \otimes B^+ & \\
A^- & ::= p^- \mid {\uparrow}A^+ \mid A^+ \multimap B^- & \\
\Gamma & ::= \cdot \mid \Gamma, A^- & \textit{(multiset)} \\
\underline{\Delta} & ::= \cdot \mid \underline{\Delta}, A^+ \mid \underline{\Delta}, A^- \mid \underline{\Delta}, [A^-] \mid \underline{\Delta}, \langle A^+ \rangle & \textit{(multiset)} \\
\underline{U} & ::= A^- \mid A^+ \mid [A^+] \mid \langle A^- \rangle &
\end{array}
$$

$\boxed{\Gamma; \underline{\Delta} \vdash \underline{U}}$

$$
\frac{\Gamma; \Delta \vdash [A^+]}{\Gamma; \Delta \vdash A^+}\ \mathit{focus}^*_R
\qquad
\frac{\Gamma; \Delta, [A^-] \vdash U}{\Gamma; \Delta, A^- \vdash U}\ \mathit{focus}^*_L
\qquad
\frac{\Gamma, A^-; \Delta, [A^-] \vdash U}{\Gamma, A^-; \Delta \vdash U}\ \mathit{copy}^*
$$

$$
\frac{\Gamma; \Delta, \langle p^+ \rangle \vdash U}{\Gamma; \Delta, p^+ \vdash U}\ \eta^+
\qquad
\frac{}{\Gamma; \langle A^+ \rangle \vdash [A^+]}\ \mathit{id}^+
\qquad
\frac{\Gamma; \Delta \vdash \langle p^- \rangle}{\Gamma; \Delta \vdash p^-}\ \eta^-
\qquad
\frac{}{\Gamma; [A^-] \vdash \langle A^- \rangle}\ \mathit{id}^-
$$

$$
\frac{\Gamma; \Delta \vdash A^+}{\Gamma; \Delta \vdash {\uparrow}A^+}\ {\uparrow}_R
\qquad
\frac{\Gamma; \Delta, A^+ \vdash U}{\Gamma; \Delta, [{\uparrow}A^+] \vdash U}\ {\uparrow}_L
\qquad
\frac{\Gamma; \Delta \vdash A^-}{\Gamma; \Delta \vdash [{\downarrow}A^-]}\ {\downarrow}_R
\qquad
\frac{\Gamma; \Delta, A^- \vdash U}{\Gamma; \Delta, {\downarrow}A^- \vdash U}\ {\downarrow}_L
$$

$$
\frac{\Gamma; \cdot \vdash A^-}{\Gamma; \cdot \vdash [!A^-]}\ !_R
\qquad
\frac{\Gamma, A^-; \Delta \vdash U}{\Gamma; \Delta, !A^- \vdash U}\ !_L
\qquad
\frac{}{\Gamma; \cdot \vdash [\mathbf{1}]}\ \mathbf{1}_R
\qquad
\frac{\Gamma; \Delta \vdash U}{\Gamma; \Delta, \mathbf{1} \vdash U}\ \mathbf{1}_L
$$

$$
\frac{\Gamma; \Delta_1 \vdash [A^+] \quad \Gamma; \Delta_2 \vdash [B^+]}{\Gamma; \Delta_1, \Delta_2 \vdash [A^+ \otimes B^+]}\ \otimes_R
\qquad
\frac{\Gamma; \Delta, A^+, B^+ \vdash U}{\Gamma; \Delta, A^+ \otimes B^+ \vdash U}\ \otimes_L
$$

$$
\frac{\Gamma; \Delta, A^+ \vdash B^-}{\Gamma; \Delta \vdash A^+ \multimap B^-}\ {\multimap}_R
\qquad
\frac{\Gamma; \Delta_1 \vdash [A^+] \quad \Gamma; \Delta_2, [B^-] \vdash U}{\Gamma; \Delta_1, \Delta_2, [A^+ \multimap B^-] \vdash U}\ {\multimap}_L
$$

Figure 2.5: Focused intuitionstic linear logic

The persistent context of a focused derivation can always be weakened by adding more persistent resources. This weakening property can be phrased as an admissible rule, which we indicate using a dashed line:

$$
\frac{\Gamma; \underline{\Delta} \vdash \underline{U}}{\Gamma, \Gamma'; \underline{\Delta} \vdash \underline{U}}\ \mathit{weaken}
$$

In developments following Pfenning's structural cut admissibility methodology [Pfe00], it is critical that the weakening theorem *does not* change the structure of proofs: that the structure of the derivation $\Gamma; \underline{\Delta} \vdash \underline{U}$ is unchanged when we weaken it to $\Gamma, \Gamma'; \underline{\Delta} \vdash \underline{U}$. It turns out that the development in this chapter does not rely on this property.

Suspended propositions ($\langle A^+ \rangle$ and $\langle A^- \rangle$) and the four rules that interact with suspended propositions (id^+, id^-, η^+, and η^-) are the main nonstandard aspect of this presentation. The η^+ and η^- rules, which allow us to stop decomposing a proposition that we are eagerly decomposing with invertible rules, are restricted to atomic propositions, and there is no other way for suspended propositions to be introduced into the context with rules. It seems reasonable to restrict the two rules that capture the identity principles, id^+ and id^-, to atomic propositions as well. However, the seemingly unnecessary generality of these two identity rules makes it much easier to establish the standard metatheory of this sequent calculus. To see why this is the case, we will turn our attention to suspended propositions and the four admissible rules (two focal substitution principles and two identity expansion principles) that interact with suspended propositions.

2.3.4 Suspended propositions

In unfocused sequent calculi, it is generally possible to restrict the id rule to atomic propositions (as shown in Figure 2.1). The general id rule, which concludes $\Gamma; A \longrightarrow A$ for all propositions A, is admissible just as the cut rule is admissible. But while the cut rule can be eliminated completely, the atomic id rule must remain. This is related to the logical interpretation of atomic propositions as stand-ins for unknown propositions. All sequent calculi, focused or unfocused, have the subformula property: every rule breaks down a proposition, either on the left or the right of the turnstile "$\vdash$", when read from bottom to top. We are unable to break down atomic propositions any further (they are unknown), thus the id rule is necessary at atomic propositions. If we substitute a concrete proposition for some atomic proposition, the structure of the proof stays exactly the same, except that instances of initial sequents become admissible instances of the identity theorem.

To my knowledge, all published proof systems for focused logic have incorporated a focused version of the id rule that also applies only to atomic propositions. This treatment is not incorrect and is obviously analogous to the id rule from the unfocused system. Nevertheless, I believe this to be a design error, and it is one that has historically made it unnecessarily difficult to prove the identity theorem for focused systems. The alternative developed in this chapter is the use of suspensions. Suspended positive propositions $\langle A^+ \rangle$ only appear in the linear context Δ, and suspended negative propositions $\langle A^- \rangle$ only appear as succedents. They are treated as stable (we never break down a suspended proposition) and are only used to immediately prove a proposition in focus with one of the identity rules id^+ or id^-. The rules id^+ and id^- are more general focused versions of the unfocused id rule. This extra generality does not influence the structure of proofs because suspended propositions can only be introduced into the context or the succedent by the η^+ and η^- rules, and those rules *are* restricted to atomic propositions.

Suspended positive propositions act much like regular variables in a natural deduction system. The positive identity rule id^+ allows us to prove any positive proposition given that the positive proposition appears suspended in the context. There is a corresponding substitution principle for focal substitutions that has a natural-deduction-like flavor: we can substitute a derivation right-focused on A^+ for a suspended positive proposition $\langle A^+ \rangle$ in a context.

Theorem 2.1 (Focal substitution (positive))**.**
If $\Gamma; \Delta \vdash [A^+]$ *and* $\Gamma; \underline{\Delta'}, \langle A^+ \rangle \vdash \underline{U}$*, then* $\Gamma; \underline{\Delta'}, \Delta \vdash \underline{U}$.

Proof. Straightforward induction over the second given derivation, as in a proof of regular substitution in a natural deduction system. If the second derivation is the axiom id^+, the result follows immediately using the first given derivation. □

As discussed above in Section 2.3.3, because we only consider focused sequents that are otherwise stable, we assume that Δ in the statement of Theorem 2.1 is stable by virtue of it appearing in the focused sequent $\Gamma; \Delta \vdash [A^+]$. The second premise $\Gamma; \underline{\Delta'}, \langle A^+ \rangle \vdash \underline{U}$, on the other hand, may be a right-focused sequent $\Gamma; \Delta', \langle A^+ \rangle \vdash [B^+]$, a left-focused sequent $\Gamma; \Delta'', [B^-], \langle A^+ \rangle \vdash U$, or an inverting sequent.

Suspended negative propositions are a bit less intuitive than suspended positive propositions. While a derivation of $\Gamma; \underline{\Delta'}, \langle A^+ \rangle \vdash \underline{U}$ is missing a premise that can be satisfied by a derivation of $\Gamma; \Delta \vdash [A^+]$, a derivation of $\Gamma; \underline{\Delta} \vdash \langle A^- \rangle$ is missing a *continuation* that can be satisfied by a derivation of $\Gamma; \Delta', [A^-] \vdash U$. The focal substitution principle, however, still takes the basic form of a substitution principle.

Theorem 2.2 (Focal substitution (negative))**.**
If $\Gamma; \underline{\Delta} \vdash \langle A^- \rangle$ *and* $\Gamma; \Delta', [A^-] \vdash U$*, then* $\Gamma; \Delta', \underline{\Delta} \vdash U$.

Proof. Straightforward induction over the *first* given derivation; if the first derivation is the axiom id^-, the result follows immediately using the second given derivation. □

Unlike cut admissibility, which we discuss in Section 2.3.6, both of the focal substitution principles are straightforward inductions over the structure of the derivation containing the suspended proposition. As an aside, when we encode the focused sequent calculus for persistent logic in LF, a suspended positive premise can be naturally encoded as a hypothetical right focus. This encoding makes the id^+ rule an instance of the hypothesis rule provided by LF and establishes Theorem 2.1 "for free" as an instance of LF substitution. This is possible to do for negative focal substitution as well, but it is counterintuitive and relies on a peculiar use of LF's uniform function space [Sim11].

The two substitution principles can be phrased as admissible rules for building derivations, like the $weaken$ rule above:

$$\frac{\Gamma; \Delta \vdash [A^+] \quad \Gamma; \underline{\Delta'}, \langle A^+ \rangle \vdash \underline{U}}{\Gamma; \underline{\Delta'}, \Delta \vdash \underline{U}}\ subst^+ \qquad \frac{\Gamma; \underline{\Delta} \vdash \langle A^- \rangle \quad \Gamma; \Delta', [A^-] \vdash U}{\Gamma; \Delta', \underline{\Delta} \vdash U}\ subst^-$$

Note the way in which these admissible substitution principles generalize the logic: $subst^+$ or $subst^-$ are the only rules we have discussed that allow us to introduce non-atomic suspended propositions, because only *atomic* suspended propositions are introduced explicitly by rules η^+ and η^-.

2.3.5 Identity expansion

Suspended propositions appear in Figure 2.5 in two places: in the identity rules, which we have just discussed and connected with the focal substitution principles, and in the rules marked η^+ and η^-, which are also the only mention of atomic propositions in the presentation. It is here that we need to make a critical shift of perspective from unfocused to focused logic. In an unfocused logic, the rules nondeterministically break down propositions, and the initial rule id puts an end to this process when an atomic proposition is reached. In a focused logic, the focus and inversion phases must break down a proposition *all the way* until a shift is reached. The two η rules are what put an end to this when an atomic proposition is reached, and they work hand-in-glove with the two id rules that allow these necessarily suspended propositions to successfully conclude a right or left focus.

Just as the id rule is a particular instance of the admissible identity sequent $\Gamma; A \longrightarrow A$ in unfocused linear logic, the atomic suspension rules η^+ and η^- are instances of an admissible *identity expansion* rule in focused linear logic:

$$\frac{\Gamma; \Delta, \langle A^+ \rangle \vdash U}{\Gamma; \Delta, A^+ \vdash U}\ \eta^+ \qquad \frac{\Gamma; \Delta \vdash \langle A^- \rangle}{\Gamma; \Delta \vdash A^-}\ \eta^-$$

In other words, the admissible identity expansion rules allow us to act as if the η^+ and η^- rules apply to *arbitrary* propositions, not just atomic propositions. The atomic propositions must be handled by an explicit rule, but the general principle is admissible.

The two admissible identity expansion rules above can be rephrased as an identity expansion theorem:

Theorem 2.3 (Identity expansion)**.**
- *If* $\Gamma; \Delta, \langle A^+ \rangle \vdash U$*, then* $\Gamma; \Delta, A^+ \vdash U$.
- *If* $\Gamma; \Delta \vdash \langle A^- \rangle$*, then* $\Gamma; \Delta \vdash A^-$.

Proof. Mutual induction over the structure of the proposition A^+ or A^-, with a critical use of focal substitution in each case.

Most of the cases of this proof are represented in Figure 2.6. The remaining case (for the multiplicative unit $\mathbf{1}$) is presented in Figure 2.7 along with the cases for the additive connectives $\mathbf{0}$, $\oplus$, $\top$, and $\&$, which are neglected elsewhere in this chapter. (Note that in Figures 2.6 and 2.7 we omit polarity annotations from propositions as they are always clear from the context.) $\square$

The admissible identity expansion rules fit with an interpretation of positive atomic propositions as stand-ins for arbitrary positive propositions and of negative atomic propositions as stand-ins for negative atomic propositions: if we substitute a proposition for some atomic proposition, all the instances of atomic suspension corresponding to that rule become admissible instances of identity expansion.

The usual identity principles are corollaries of identity expansion:

$$\cfrac{\cfrac{\cfrac{}{\Gamma; \langle A^+ \rangle \vdash [A^+]}\ id^+}{\Gamma; \langle A^+ \rangle \vdash A^+}\ focus_R}{\Gamma; A^+ \vdash A^+}\ \eta^+ \qquad \cfrac{\cfrac{\cfrac{}{\Gamma; [A^-] \vdash \langle A^- \rangle}\ id^-}{\Gamma; A^- \vdash \langle A^- \rangle}\ focus_L}{\Gamma; A^- \vdash A^-}\ \eta^-$$

$$\dfrac{\overset{\mathcal{D}}{\Gamma;\Delta,\langle{\downarrow}A\rangle\vdash U}}{\Gamma;\Delta,{\downarrow}A\vdash U}\,\eta^+ \quad\Longrightarrow\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma;[A]\vdash\langle A\rangle}\,\mathit{id}^-}{\Gamma;A\vdash\langle A\rangle}\,\mathit{focus}_L}{\Gamma;A\vdash A}\,\eta^-}{\Gamma;A\vdash[{\downarrow}A]}\,{\downarrow}_R \qquad \overset{\mathcal{D}}{\Gamma;\Delta,\langle{\downarrow}A\rangle\vdash U}}{\Gamma;\Delta,A\vdash U}\,\mathit{subst}^+}{\Gamma;\Delta,{\downarrow}A\vdash U}\,{\downarrow}_L$$

$$\dfrac{\overset{\mathcal{D}}{\Gamma;\Delta,\langle !A\rangle\vdash U}}{\Gamma;\Delta,!A\vdash U}\,\eta^+ \quad\Longrightarrow\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma,A;[A]\vdash\langle A\rangle}\,\mathit{id}^-}{\Gamma,A;\cdot\vdash\langle A\rangle}\,\mathit{copy}}{\Gamma,A;\cdot\vdash A}\,\eta^-}{\Gamma,A;\cdot\vdash[!A]}\,!_R \qquad \dfrac{\overset{\mathcal{D}}{\Gamma;\Delta,\langle !A\rangle\vdash U}}{\Gamma,A;\Delta,\langle !A\rangle\vdash U}\,\mathit{weaken}}{\Gamma,A;\Delta\vdash U}\,\mathit{subst}^+}{\Gamma;\Delta,!A\vdash U}\,!_L$$

$$\dfrac{\overset{\mathcal{D}}{\Gamma;\Delta,\langle A\otimes B\rangle\vdash U}}{\Gamma;\Delta,A\otimes B\vdash U}\,\eta^+ \quad\Longrightarrow\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma;\langle A\rangle\vdash[A]}\,\mathit{id}^+ \quad \dfrac{}{\Gamma;\langle B\rangle\vdash[B]}\,\mathit{id}^+}{\Gamma;\langle A\rangle,\langle B\rangle\vdash[A\otimes B]}\,\otimes_R \qquad \overset{\mathcal{D}}{\Gamma;\Delta,\langle A\otimes B\rangle\vdash U}}{\Gamma;\Delta,\langle A\rangle,\langle B\rangle\vdash U}\,\mathit{subst}^+}{\Gamma;\Delta,\langle A\rangle,B\vdash U}\,\eta^+}{\Gamma;\Delta,A,B\vdash U}\,\eta^+}{\Gamma;\Delta,A\otimes B\vdash U}\,\otimes_L$$

$$\dfrac{\overset{\mathcal{D}}{\Gamma;\Delta\vdash\langle{\uparrow}A\rangle}}{\Gamma;\Delta\vdash{\uparrow}A}\,\eta^- \quad\Longrightarrow\quad \dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma;\Delta\vdash\langle{\uparrow}A\rangle} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma;\langle A\rangle\vdash[A]}\,\mathit{id}^+}{\Gamma;\langle A\rangle\vdash A}\,\mathit{focus}_R}{\Gamma;A\vdash A}\,\eta^+}{\Gamma;[{\uparrow}A]\vdash A}\,{\uparrow}_L}{\Gamma;\Delta\vdash A}\,\mathit{subst}^-}{\Gamma;\Delta\vdash{\uparrow}A}\,{\uparrow}_R$$

$$\dfrac{\overset{\mathcal{D}}{\Gamma;\Delta\vdash\langle A\multimap B\rangle}}{\Gamma;\Delta\vdash A\multimap B}\,\eta^- \quad\Longrightarrow\quad \dfrac{\dfrac{\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma;\Delta\vdash\langle A\multimap B\rangle} \qquad \dfrac{\dfrac{}{\Gamma;\langle A\rangle\vdash[A]}\,\mathit{id}^+ \quad \dfrac{}{\Gamma;[B]\vdash\langle B\rangle}\,\mathit{id}^-}{\Gamma;\langle A\rangle,[A\multimap B]\vdash\langle B\rangle}\,\multimap_L}{\Gamma;\Delta,\langle A\rangle\vdash\langle B\rangle}\,\mathit{subst}^-}{\Gamma;\Delta,\langle A\rangle\vdash B}\,\eta^-}{\Gamma;\Delta,A\vdash B}\,\eta^+}{\Gamma;\Delta\vdash A\multimap B}\,\multimap_R$$

Figure 2.6: Identity expansion – restricting η^+ and η^- to atomic propositions

$$\dfrac{\overset{\mathcal{D}}{\Gamma; \Delta, \langle \mathbf{1} \rangle \vdash U}}{\Gamma; \Delta, \mathbf{1} \vdash U}\,\eta^+ \quad\Longrightarrow\quad \dfrac{\dfrac{\dfrac{}{\Gamma; \cdot \vdash [\mathbf{1}]}\,\mathbf{1}_R \qquad \overset{\mathcal{D}}{\Gamma; \Delta, \langle \mathbf{1} \rangle \vdash U}}{\Gamma; \Delta \vdash U}\,\mathit{subst}^+}{\Gamma; \Delta, \mathbf{1} \vdash U}\,\mathbf{1}_L$$

$$\dfrac{\overset{\mathcal{D}}{\Gamma; \Delta, \langle \mathbf{0} \rangle \vdash U}}{\Gamma; \Delta, \mathbf{0} \vdash U}\,\eta^+ \quad\Longrightarrow\quad \dfrac{}{\Gamma; \Delta, \mathbf{0} \vdash U}\,\mathbf{0}_L$$

$$\dfrac{\overset{\mathcal{D}}{\Gamma; \Delta, \langle A \oplus B \rangle \vdash U}}{\Gamma; \Delta, A \oplus B \vdash U}\,\eta^+ \quad\Longrightarrow\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma; \langle A \rangle \vdash [A]}\,\mathit{id}^+}{\Gamma; \Delta, \langle A \rangle \vdash [A \oplus B]}\,\oplus_{R1} \qquad \overset{\mathcal{D}}{\Gamma; \Delta, \langle A \oplus B \rangle \vdash U}}{\Gamma; \Delta, \langle A \rangle \vdash U}\,\mathit{subst}^+}{\Gamma; \Delta, A \vdash U}\,\eta^+ \qquad \overset{\vdots}{\Gamma; \Delta, B \vdash U}}{\Gamma; \Delta, A \oplus B \vdash U}\,\oplus_L$$

$$\dfrac{\overset{\mathcal{D}}{\Gamma; \Delta \vdash \langle \top \rangle}}{\Gamma; \Delta \vdash \top}\,\eta^- \quad\Longrightarrow\quad \dfrac{}{\Gamma; \Delta \vdash \top}\,\top_R$$

$$\dfrac{\overset{\mathcal{D}}{\Gamma; \Delta \vdash \langle A \mathbin{\&} B \rangle}}{\Gamma; \Delta \vdash A \mathbin{\&} B}\,\eta^- \quad\Longrightarrow\quad \dfrac{\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma; \Delta \vdash \langle A \mathbin{\&} B \rangle} \qquad \dfrac{\dfrac{}{\Gamma; [A] \vdash \langle A \rangle}\,\mathit{id}^-}{\Gamma; [A \mathbin{\&} B] \vdash \langle A \rangle}\,\&_{L1}}{\Gamma; \Delta \vdash \langle A \rangle}\,\mathit{subst}^-}{\Gamma; \Delta \vdash A}\,\eta^- \qquad \overset{\vdots}{\Gamma; \Delta \vdash B}}{\Gamma; \Delta \vdash A \mathbin{\&} B}\,\&_R$$

Figure 2.7: Identity expansion for units and additive connectives

2.3.6 Cut admissibility

Cut admissibility, Theorem 2.4 below, mostly follows the well-worn contours of a structural cut admissibility argument [Pfe00]. A slight inelegance of the proof given here is that some very similar cases must be considered more than once in different parts of the proof. The right commutative cases – cases in which the last rule in the second given derivation is an invertible rule that is not decomposing the principal cut formula A^+ – must be repeated in parts 1 and 4, for instance. (Pfenning's classification of the cases of cut admissibility into principal, left commutative, and right commutative cuts is discussed in Section 3.4.) In addition to this duplication, the proof of part 4 is almost identical in form to the proof of part 5. The proof of cut admissibility in the next chapter will eliminate both forms of duplication.

The most important caveat about cut admissibility is that it is only applicable in the absence of any non-atomic suspended propositions. If we did not make this restriction, then in Theorem 2.4, part 1, we might encounter a derivation of $\Gamma; \langle A \otimes B \rangle \vdash [A \otimes B]$ that concludes with id^+ being cut into the derivation

$$\dfrac{\overset{\mathcal{E}}{\Gamma; \Delta', A, B \vdash U}}{\Gamma; \Delta', A \otimes B \vdash U} \otimes_R$$

in which case there is no clear way to proceed and prove $\Gamma; \Delta', \langle A \otimes B \rangle \vdash U$.

Theorem 2.4 (Cut admissibility)**.** *For all Γ, A^+, A^-, Δ, Δ', and U that do not contain any non-atomic suspended propositions:*

1. *If $\Gamma; \Delta \vdash [A^+]$ and $\Gamma; \Delta', A^+ \vdash U$ (where Δ is stable), then $\Gamma; \Delta', \Delta \vdash U$.*
2. *If $\Gamma; \Delta \vdash A^-$ and $\Gamma; \Delta', [A^-] \vdash U$ (where Δ, Δ', and U are stable), then $\Gamma; \Delta', \Delta \vdash U$.*
3. *If $\Gamma; \underline{\Delta} \vdash A^+$ and $\Gamma; \Delta', A^+ \vdash U$, (where Δ' and U are stable), then $\Gamma; \Delta', \underline{\Delta} \vdash U$.*
4. *If $\Gamma; \Delta \vdash A^-$ and $\Gamma; \underline{\Delta'}, A^- \vdash \underline{U}$, (where Δ is stable), then $\Gamma; \underline{\Delta'}, \Delta \vdash \underline{U}$.*
5. *If $\Gamma; \cdot \vdash A^-$ and $\Gamma, A^-; \underline{\Delta'} \vdash \underline{U}$, then $\Gamma; \underline{\Delta'} \vdash \underline{U}$.*

Parts 1 and 2 are where most of the action happens, but there is a sense in which the *necessary* cut admissibility property is contained in structure of parts 3, 4, and 5 – these are the cases used to prove the completeness of focusing (Theorem 2.6). The discrepancy between the stability restrictions demanded for part 1 and part 2 is discussed below; this peculiarity is justified by the fact that these two parts need only be general enough to prove parts 3, 4, and 5.

Proof. The proof is by induction: in each invocation of the induction hypothesis, either the principal cut formula A^+ or A^- gets smaller or else it stays the same and the "part size" (1-5) gets smaller. When the principal cut formula and the part size remain the same, either the first given derivation gets smaller (part 3) or the second given derivation gets smaller (parts 1, 4 and 5).

This termination argument is a refinement of the standard structural termination argument for cut admissibility in unfocused logics [Pfe00] – in part 3, we don't need to know that the second given derivation stays the same size, and in parts 1, 4, and 5 we don't need to know that the first given derivation stays the same size. This refined termination argument is the reason that we do not need to prove that admissible weakening preserves the structure of proofs.

We schematically present one or two illustrative cases for each part of the proof.

Part 1 (positive principal cuts, right commutative cuts)

(Δ_1, Δ_2 stable are stable by assumption)

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\Gamma; \Delta_1 \vdash [A_1^+]} \quad \overset{\mathcal{D}_2}{\Gamma; \Delta_2 \vdash [A_2^+]}}{\Gamma; \Delta_1, \Delta_2 \vdash [A_1^+ \otimes A_2^+]} \otimes_R \quad \dfrac{\overset{\mathcal{E}'}{\Gamma; \Delta', A_1^+, A_2^+ \vdash U}}{\Gamma; \Delta', A_1^+ \otimes A_2^+ \vdash U} \otimes_L}{\Gamma; \Delta', \Delta_1, \Delta_2 \vdash U} \; cut(1)$$

$$\Longrightarrow \quad \dfrac{\overset{\mathcal{D}_1}{\Gamma; \Delta_1 \vdash [A_1^+]} \quad \dfrac{\overset{\mathcal{D}_2}{\Gamma; \Delta_2 \vdash [A_2^+]} \quad \overset{\mathcal{E}'}{\Gamma; \Delta', A_1^+, A_2^+ \vdash U}}{\Gamma; \Delta', A_1^+, \Delta_2 \vdash U} \; cut(1)}{\Gamma; \Delta', \Delta_1, \Delta_2 \vdash U} \; cut(1)$$

(Δ is stable by assumption)

$$\dfrac{\overset{\mathcal{D}}{\Gamma;\Delta\vdash[A^+]} \qquad \dfrac{\overset{\mathcal{E}'}{\Gamma;\Delta',B_1^+,B_2^+,A^+\vdash U}}{\Gamma;\Delta',B_1^+\otimes B_2^+,A^+\vdash U}\,\otimes_L}{\Gamma;\Delta',B_1^+\otimes B_2^+,\Delta\vdash U}\,\mathit{cut}(1) \quad\Longrightarrow\quad \dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma;\Delta\vdash[A^+]}\qquad \overset{\mathcal{E}'}{\Gamma;\Delta',B_1^+,B_2^+,A^+\vdash U}}{\Gamma;\Delta',B_1^+,B_2^+,\Delta\vdash U}\,\mathit{cut}(1)}{\Gamma;\Delta',B_1^+\otimes B_2^+,\Delta\vdash U}\,\otimes_L$$

Part 2 (negative principal cuts)

(Δ, Δ', Δ'_A, and U are stable by assumption)

$$\dfrac{\dfrac{\overset{\mathcal{D}'}{\Gamma;\Delta,A_1^+\vdash A_2^-}}{\Gamma;\Delta\vdash A_1^+\multimap A_2^-}\,\multimap_R \qquad \dfrac{\overset{\mathcal{E}_1}{\Gamma;\Delta'_A\vdash[A_1^+]}\qquad \overset{\mathcal{E}_2}{\Gamma;\Delta',[A_2^-]\vdash U}}{\Gamma;\Delta',\Delta'_A,[A_1^+\multimap A_2^-]\vdash U}\,\multimap_L}{\Gamma;\Delta',\Delta'_A,\Delta\vdash U}\,\mathit{cut}(2)$$

$$\Longrightarrow\quad \dfrac{\dfrac{\overset{\mathcal{E}_1}{\Gamma;\Delta'_A\vdash[A_1^+]}\qquad \overset{\mathcal{D}'}{\Gamma;\Delta,A_1^+\vdash A_2^-}}{\Gamma;\Delta'_A,\Delta\vdash A_2^-}\,\mathit{cut}(1) \qquad \overset{\mathcal{E}_2}{\Gamma;\Delta',[A_2^-]\vdash U}}{\Gamma;\Delta',\Delta'_A,\Delta\vdash U}\,\mathit{cut}(2)$$

Part 3 (left commutative cuts)

(Δ' and U are stable by assumption, Δ is stable by the side condition on rule focus_R)

$$\dfrac{\dfrac{\overset{\mathcal{D}'}{\Gamma;\Delta\vdash[A^+]}}{\Gamma;\Delta\vdash A^+}\,\mathit{focus}_R \qquad \overset{\mathcal{E}}{\Gamma;\Delta',A^+\vdash U}}{\Gamma;\Delta',\Delta\vdash U}\,\mathit{cut}(3) \quad\Longrightarrow\quad \dfrac{\overset{\mathcal{D}'}{\Gamma;\Delta\vdash[A^+]}\qquad \overset{\mathcal{E}}{\Gamma;\Delta',A^+\vdash U}}{\Gamma;\Delta',\Delta\vdash U}\,\mathit{cut}(1)$$

(Δ' and U are stable by assumption)

$$\dfrac{\dfrac{\overset{\mathcal{D}'}{\Gamma;\Delta,B_1^+,B_2^+\vdash A^+}}{\Gamma;\Delta,B_1^+\otimes B_2^+\vdash A^+}\,\otimes_L \qquad \overset{\mathcal{E}}{\Gamma;\Delta',A^+\vdash U}}{\Gamma;\Delta',\Delta,B_1^+\otimes B_2^+\vdash A^+}\,\mathit{cut}(3) \quad\Longrightarrow\quad \dfrac{\dfrac{\overset{\mathcal{D}'}{\Gamma;\Delta,B_1^+,B_2^+\vdash A^+}\qquad \overset{\mathcal{E}}{\Gamma;\Delta',A^+\vdash U}}{\Gamma;\Delta',\Delta,B_1^+,B_2^+\vdash A^+}\,\mathit{cut}(3)}{\Gamma;\Delta',\Delta,B_1^+\otimes B_2^+\vdash A^+}\,\otimes_L$$

$$
\begin{array}{rcl|rcl|rcl}
(\Gamma)^\circ & & & (\underline{\Delta})^\circ & & & (\underline{U})^\circ & & \\
(\cdot)^\circ & = & \cdot & (\cdot)^\circ & = & \cdot & (A^-)^\circ & = & (A^-)^\circ \\
(\Gamma, A^-)^\circ & = & (\Gamma)^\circ, (A^-)^\circ & (\Delta, A^+)^\circ & = & (\Delta)^\circ, (A^+)^\circ & (A^+)^\circ & = & (A^+)^\circ \\
 & & & (\Delta, A^-)^\circ & = & (\Delta)^\circ, (A^-)^\circ & ([A^+])^\circ & = & (A^+)^\circ \\
 & & & (\Delta, [A^-])^\circ & = & (\Delta)^\circ, (A^-)^\circ & (\langle p^- \rangle)^\circ & = & p^- \\
 & & & (\Delta, \langle p^+ \rangle)^\circ & = & (\Delta)^\circ, p^+ & & &
\end{array}
$$

Figure 2.8: Lifting erasure and polarization (Figure 2.3) to contexts and succedents

Part 4 (right commutative cuts)

(Δ is stable by assumption, Δ' and U are stable by the side condition on rule focus_R)

$$
\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma; \Delta \vdash A^-\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}'\\ \Gamma; \Delta', [A^-] \vdash U\end{array}}{\Gamma; \Delta', A^- \vdash U}\ \mathit{focus}_R}{\Gamma; \Delta', \Delta \vdash U}\ \mathit{cut}(4)
\quad \Longrightarrow \quad
\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma; \Delta \vdash A^-\end{array} \quad \begin{array}{c}\mathcal{E}'\\ \Gamma; \Delta', [A^-] \vdash U\end{array}}{\Gamma; \Delta', \Delta \vdash U}\ \mathit{cut}(2)
$$

Part 5 (persistent right commutative cuts)

$$
\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma; \cdot \vdash A^-\end{array} \quad \dfrac{\begin{array}{c}\mathcal{E}'\\ \Gamma, A^-; \cdot \vdash B^-\end{array}}{\Gamma, A^-; \cdot \vdash [!B^-]}\ !_R}{\Gamma; \cdot \vdash [!B^-]}\ \mathit{cut}(5)
\quad \Longrightarrow \quad
\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma; \cdot \vdash A^-\end{array} \quad \begin{array}{c}\mathcal{E}'\\ \Gamma, A^-; \cdot \vdash B^-\end{array}}{\Gamma; \cdot \vdash B^-}\ \mathit{cut}(5)}{\Gamma; \cdot \vdash [!B^-]}\ !_R
$$

All the other cases follow the same pattern. □

As noted above, there is a notable asymmetry between part 1 of the theorem, which does not require stability of Δ' and U in the second given derivation $\Gamma; \Delta', A^+ \vdash U$, and part 2 of the theorem, which does require stability of Δ in the first given derivation $\Gamma; \Delta \vdash A^-$. The theorem would still hold for non-stable Δ, but we do not need the more general theorem, and the less general theorem is easier to prove – it allows us to avoid duplicating the left commutative cuts between parts 2 and 3. On the other hand, we cannot make the theorem more specific, imposing extra stability conditions on part 1, without fixing the order in which invertible rules are applied. Fixing the order in which invertible rules are applied has some other advantages as well; this is a point we will return to in Section 2.3.8.

2.3.7 Correctness of focusing

Now we will prove the correctness property for the focused, polarized logic that we discussed in Section 2.3.1: that there is an unfocused derivation of $(A^+)^\circ$ or $(A^-)^\circ$ if and only if there is a focused derivation of A^+ or A^-. The proof requires us to lift our erasure function to contexts and succedents, which is done in Figure 2.8. Note that erasure is only defined on focused sequents

$\Gamma; \underline{\Delta} \vdash \underline{U}$ when all suspended propositions are atomic. We are justified in making this restriction because non-atomic suspended propositions cannot arise in the process of proving a proposition A^+ or A^- in an empty context, and we are required to make this restriction due to the analogous restrictions on cut admissibility (Theorem 2.4).

Theorems 2.5 and 2.6 therefore implicitly carry the same extra condition that we put on the cut admissibility theorem: that $\underline{\Delta}$ and $\underline{U}$ must contain only atomic suspended propositions.

Theorem 2.5 (Soundness of focusing)**.** *If* $\Gamma; \underline{\Delta} \vdash \underline{U}$*, then* $\Gamma^\circ; \underline{\Delta}^\circ \longrightarrow \underline{U}^\circ$.

Proof. By straightforward induction on the given derivation; in each case, the result either follows directly by invoking the induction hypothesis (in the case of rules like $\uparrow_R$) or by invoking the induction hypothesis and applying one rule from Figure 2.1 (in the case of rules like $\otimes_R$). □

Theorem 2.6 (Completeness of focusing)**.** *If* $\Gamma^\circ; \Delta^\circ \longrightarrow U^\circ$*, where* Δ *and* U *are stable, then* $\Gamma; \Delta \vdash U$.

Proof. By induction on the first given derivation. Each rule in the unfocused system (Figure 2.1) corresponds to one *unfocused admissibility lemma*, plus some extra steps.

These extra steps arise are due to the generality of erasure. If we know that $!A = (C^+)^\circ$ (as in the case for $!_R$ below), then by case analysis on the structure of C^+, C^+ must be either $!B^-$ (for some B^-) or $\downarrow C_1^-$ (for some C_1^-). In the latter case, by further case analysis on C_1^- we can see that C_1^- must equal $\uparrow C_2^+$ (for some C_2^+). But then C_2^+ can be either $!B_2^-$ or $\downarrow C_3^-$; in the latter case $C^+ = \downarrow\uparrow\downarrow C_3^-$, and this can go on arbitrarily long (but not forever, because C^- is a finite term). So we say that, by induction on the structure of C^+, there exists an A^- such that $C^+ = \downarrow\uparrow \ldots \downarrow\uparrow !A^-$ and $A = (A^-)^\circ$. Depending on the case, we then repeatedly apply either the $\uparrow\downarrow_R$ rule or the $\downarrow\uparrow_L$ rule, both of which are derived below, to eliminate all the extra shifts. (Zero or more instances of a rule are indicated by a double-ruled inference rule.)

$$\dfrac{\Gamma; \Delta \vdash A^+}{\Gamma; \Delta \vdash \downarrow\uparrow A^+}\ \downarrow\uparrow_R = \dfrac{\dfrac{\dfrac{\Gamma; \Delta \vdash A^+}{\Gamma; \Delta \vdash \uparrow A^+}\ \uparrow_R}{\Gamma; \Delta \vdash [\downarrow\uparrow A^+]}\ \downarrow_R}{\Gamma; \Delta \vdash \downarrow\uparrow A^+}\ \mathit{focus}_R \qquad \dfrac{\Gamma; \Delta, A^- \vdash U}{\Gamma; \Delta, \uparrow\downarrow A^- \vdash U}\ \uparrow\downarrow_L = \dfrac{\dfrac{\dfrac{\Gamma; \Delta, A^- \vdash U}{\Gamma; \Delta, \downarrow A^- \vdash U}\ \downarrow_R}{\Gamma; \Delta, [\uparrow\downarrow A^-] \vdash U}\ \uparrow_L}{\Gamma; \Delta, \uparrow\downarrow A^- \vdash U}\ \mathit{focus}_L$$

We will describe a few cases to illustrate how unfocused admissibility lemmas work.

Rule *copy*: We are given $\Gamma^\circ, A; \Delta^\circ, A \longrightarrow U^\circ$, which is used to derive $\Gamma^\circ, A; \Delta^\circ \longrightarrow U^\circ$. We know $A = (A^-)^\circ$. By the induction hypothesis, we have $\Gamma, A^-; \Delta, A^- \vdash U$, and we conclude with the unfocused admissibility lemma $copy_u$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, A^-; [A^-] \vdash \langle A^- \rangle}\ id^-}{\Gamma, A^-; \cdot \vdash \langle A^- \rangle}\ copy}{\Gamma, A^-; \cdot \vdash A^-}\ \eta^- \qquad \Gamma, A^-; \Delta, A^- \vdash U}{\Gamma, A^-; \Delta \vdash U}\ cut(5)$$

Rule $!_L$: We are given $\Gamma^\circ, A; \Delta^\circ \longrightarrow U^\circ$, which is used to derive $\Gamma^\circ; \Delta^\circ, !A \longrightarrow U^\circ$. We know $!A = (C^-)^\circ$; by induction on the structure of C^- there exists A^- such that $C^- =$

$\uparrow\downarrow\ldots\downarrow\uparrow !A^-$. By the induction hypothesis, we have $\Gamma, A^-; \Delta \vdash U$, and we conclude by the unfocused admissibility lemma $!_{uL}$, which is derivable:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, A^-; \Delta \vdash U}{\Gamma; \Delta, !A^- \vdash U}\,!_L}{\Gamma; \Delta, [\uparrow !A^-] \vdash U}\,\uparrow_L}{\Gamma; \Delta, \uparrow !A^- \vdash U}\,\mathit{focus}_L}{\Gamma; \Delta, \uparrow\downarrow\ldots\downarrow\uparrow !A^- \vdash U}\,\uparrow\downarrow_L$$

Rule $!_R$: We are given $\Gamma^\circ; \cdot \longrightarrow A$, which is used to derive $\Gamma^\circ; \cdot \longrightarrow !A$. We know $!A = (C^+)^\circ$; by induction on the structure of C^+ there exists A^- such that $C^+ = \downarrow\uparrow\ldots\downarrow\uparrow !A^-$. By the induction hypothesis, we have $\Gamma; \cdot \vdash \downarrow A^-$, and we conclude by the unfocused admissibility lemma $!_{uR}$:

$$\dfrac{\dfrac{\dfrac{\Gamma; \cdot \vdash \downarrow A^-}{\Gamma; \cdot \vdash \uparrow\downarrow A^-}\,\uparrow_R \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \uparrow\downarrow A^-; [A^-] \vdash \langle A^- \rangle}\,id^-}{\Gamma, \uparrow\downarrow A^-; A^- \vdash \langle A^- \rangle}\,\mathit{focus}_L}{\Gamma, \uparrow\downarrow A^-; \downarrow A^- \vdash \langle A^- \rangle}\,\downarrow_L}{\Gamma, \uparrow\downarrow A^-; [\uparrow\downarrow A^-] \vdash \langle A^- \rangle}\,\uparrow_L}{\Gamma, \uparrow\downarrow A^-; \cdot \vdash \langle A^- \rangle}\,\mathit{copy}}{\Gamma, \uparrow\downarrow A^-; \cdot \vdash A^-}\,\eta^-}{\Gamma, \uparrow\downarrow A^-; \cdot \vdash [!A^-]}\,!_R}{\Gamma, \uparrow\downarrow A^-; \cdot \vdash !A^-}\,\mathit{focus}_R}{\Gamma; \cdot \vdash !A^-}\,\mathit{cut}(5)}{\Gamma; \cdot \vdash \downarrow\uparrow\ldots\downarrow\uparrow !A^-}\,\downarrow\uparrow_R$$

Rule $\multimap_L$: We are given $\Gamma^\circ; \Delta_A^\circ \longrightarrow A$ and $\Gamma^\circ; \Delta^\circ, B \longrightarrow U^\circ$, which are used to derive $\Gamma^\circ; \Delta_A^\circ, \Delta^\circ, A \multimap B \longrightarrow U$. We know $A \multimap B = (C^-)^\circ$; by induction on the structure of C^- there exist A^+ and B^- such that $A = (A^+)^\circ$, $B = (B^-)^\circ$, and $C^- = \uparrow\downarrow\ldots\uparrow\downarrow(A^+ \multimap B^-)$. By the induction hypothesis, we have $\Gamma; \Delta_A \vdash A^+$ and $\Gamma; \Delta, B^- \vdash U$, and we conclude by the unfocused admissibility lemma $\multimap_{uL}$:

$$\dfrac{\dfrac{\dfrac{\Gamma; \Delta_A \vdash A^+ \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma; \langle A^+ \rangle \vdash [A^+]}\,id^+ \qquad \dfrac{}{\Gamma; [B^-] \vdash \langle B^- \rangle}\,id^-}{\Gamma; \langle A^+ \rangle, [A^+ \multimap B^-] \vdash \langle B^- \rangle}\,\multimap_L}{\Gamma; \langle A^+ \rangle, A^+ \multimap B^- \vdash \langle B^- \rangle}\,\mathit{focus}_L}{\Gamma; \langle A^+ \rangle, A^+ \multimap B^- \vdash B^-}\,\eta^-}{\Gamma; \langle A^+ \rangle, A^+ \multimap B^- \vdash [\downarrow B^-]}\,\downarrow_R}{\Gamma; \langle A^+ \rangle, A^+ \multimap B^- \vdash \downarrow B^-}\,\mathit{focus}_R}{\Gamma; A^+, A^+ \multimap B^- \vdash \downarrow B^-}\,\eta^+}{\Gamma; \Delta_A, A^+ \multimap B^- \vdash \downarrow B^-}\,\mathit{cut}(3) \qquad \dfrac{\Gamma; \Delta, B^- \vdash U}{\Gamma; \Delta, \downarrow B^- \vdash U}\,B^-}{\Gamma; \Delta_A, \Delta, A^+ \multimap B^- \vdash U}\,\mathit{cut}(3)}{\Gamma; \Delta_A, \Delta, \uparrow\downarrow\ldots\uparrow\downarrow(A^+ \multimap B^-) \vdash U}\,\uparrow\downarrow_L$$

Rule $\multimap_R$: We are given $\Gamma^\circ; \Delta^\circ, A \longrightarrow B$, which is used to derive $\Gamma^\circ; \Delta^\circ \longrightarrow A \multimap B$. We know $A \multimap B = (C^+)^\circ$; by induction on the structure of C^+ there exist A^+ and B^- such that $A = (A^+)^\circ$, $B = (B^-)^\circ$, and $C^+ = {\downarrow}{\uparrow}\ldots{\uparrow}{\downarrow}(A^+ \multimap B^-)$. By the induction hypothesis, we have $\Gamma; \Delta, {\uparrow}A^+ \vdash {\downarrow}B^+$, and we conclude by the unfocused admissibility lemma $\multimap_{uR}$:

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\Gamma;\Delta,{\uparrow}A^+ \vdash {\downarrow}B^-}{\Gamma;\Delta,{\uparrow}A^+ \vdash {\uparrow}{\downarrow}B^-}\,{\uparrow}_R}{\Gamma;\Delta,{\downarrow}{\uparrow}A^+ \vdash {\uparrow}{\downarrow}B^-}\,{\downarrow}_L}{\Gamma;\Delta \vdash {\downarrow}{\uparrow}A^+ \multimap {\uparrow}{\downarrow}B^-}\,{\multimap}_R
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma;\langle A^+\rangle \vdash [A^+]}\,id^+}{\Gamma;\langle A^+\rangle \vdash A^+}\,focus_R}{\Gamma;\langle A^+\rangle \vdash {\uparrow}A^+}\,{\uparrow}_R}{\Gamma;\langle A^+\rangle \vdash [{\downarrow}{\uparrow}A^+]}\,{\downarrow}_R
\quad
\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma;[B^-] \vdash \langle B^-\rangle}\,id^-}{\Gamma;B^- \vdash \langle B^-\rangle}\,focus_L}{\Gamma;{\downarrow}B^- \vdash \langle B^-\rangle}\,{\downarrow}_L}{\Gamma;[{\uparrow}{\downarrow}B^-] \vdash \langle B^-\rangle}\,{\uparrow}_L
}{\Gamma;[{\downarrow}{\uparrow}A^+ \multimap {\uparrow}{\downarrow}B^-],\langle A^+\rangle \vdash \langle B^-\rangle}\,{\multimap}_L
}{\Gamma;{\downarrow}{\uparrow}A^+ \multimap {\uparrow}{\downarrow}B^-,\langle A^+\rangle \vdash \langle B^-\rangle}\,focus_L
}{\Gamma;{\downarrow}{\uparrow}A^+ \multimap {\uparrow}{\downarrow}B^-,\langle A^+\rangle \vdash B^-}\,\eta^-
}{\Gamma;{\downarrow}{\uparrow}A^+ \multimap {\uparrow}{\downarrow}B^-,A^+ \vdash B^-}\,\eta^+
}{\Gamma;{\downarrow}{\uparrow}A^+ \multimap {\uparrow}{\downarrow}B^- \vdash A^+ \multimap B^-}\,{\multimap}_R
}{\Gamma;{\downarrow}{\uparrow}A^+ \multimap {\uparrow}{\downarrow}B^- \vdash [{\downarrow}(A^+ \multimap B^-)]}\,{\downarrow}_R
}{\Gamma;{\downarrow}{\uparrow}A^+ \multimap {\uparrow}{\downarrow}B^- \vdash {\downarrow}(A^+ \multimap B^-)}\,focus_R
}{
\dfrac{\Gamma;\Delta \vdash {\downarrow}(A^+ \multimap B^-)}{\Gamma;\Delta \vdash {\downarrow}{\uparrow}\ldots{\uparrow}{\downarrow}(A^+ \multimap B^-)}\,{\downarrow}{\uparrow}_R
}\,cut(4)
$$

All the other cases follow the same pattern. $\square$

2.3.8 Confluent versus fixed inversion

A salient feature of this presentation of focusing is that invertible, non-focused rules need not be applied in any particular order. Therefore, the last step in a proof of $\Gamma; \Delta, A \otimes B, \mathbf{1}, !C \vdash D \multimap E$ could be $\otimes_L$, $\mathbf{1}_L$ $!_L$, or $\multimap_R$. The style is exemplified by Liang and Miller's LJF [LM09], and the confluent presentation in this chapter is closely faithful to Pfenning's course notes on linear logic [Pfe12].

Allowing for this inessential nondeterminism simplifies the presentation a bit, but it also gets in the way of effective proof search and canonical derivations if we do not address it in some way. The different possibilities for addressing this nondeterminism within an inversion phase echo the discussion of nondeterminism in LF from the beginning of the chapter. We can, as suggested in that discussion, declare that all proofs which differ only by the order of their invertible, non-focused rules be treated as equivalent. It is possible to establish that all possible inversion orderings will lead to the same set of stable sequents, which lets us know that all of these reorderings do not fundamentally change the structure of the rest of the proof. This property already seems to be necessary to prove *unfocused cut* as expressed by this admissible rule:

$$
\dfrac{\Gamma; \Delta \vdash A \quad \Gamma; \Delta', A \vdash U}{\Gamma; \Delta', \Delta \vdash U}\ cut
$$

(where A is A^+ or A^- and Δ, Δ', and U contain no focus but may not be stable). If A is A^+, proving the admissibility of this rule involves permuting invertible rules in the second given derivation, $\Gamma; \Delta', A^+ \vdash U$, until A^+ is the only unstable part of the second sequent, at which point part 3 of Theorem 2.4 applies. Similarly, if A is A^-, we must permute invertible rules in the first given derivation until A^- is the only unstable part of the first sequent, at which point part 4 of Theorem 2.4 applies.

By proving and using this more general cut property, it would be possible to prove a more general completeness theorem: if $\Gamma^\circ; \Delta^\circ \longrightarrow U^\circ$, then $\Gamma; \Delta \vdash U$ (Theorem 2.6 as stated also requires that Δ and U be stable). The cases of this new theorem corresponding to the unfocused rules $!_R$, $\multimap_R$, and $\otimes_L$, which required the use of doubly-shifted side derivations in our presentation, are trivial in this modified presentation. Unfortunately, the proof of unfocused cut, while simple, is tedious and long. Gentzen's original proof of cut admissibility [Gen35] and Pfenning's mechanization [Pfe00] both scale linearly with the number of connectives and rules in the logic; the proofs of identity expansion, cut admissibility, soundness of focusing, and completeness of focusing presented in this chapter do too. There is no known proof of the unfocused admissibly of the rule cut above that scales linearly in this way: all known proofs grow quadratically with the number of connectives and rules in the logic.

Once we equate all proofs that differ only on the order in which inference rules are applied within an inversion phase, we can pick some member of each equivalence class to serve as a canonical representative; this will suffice to solve the problems with proof search, as we can search for the canonical representatives of focused proofs rather than searching within the larger set of all focused proofs. The most common canonical representatives force invertible rules to decompose propositions in a depth-first ordering.

Then, reminiscent of the move from LF to Canonical LF, the logic itself can be restricted so that only the canonical representatives are admitted. The most convenient way of forcing a left-most, depth-first ordering is to isolate the invertible propositions (A^+ on the left and A^- on the right) in separate, ordered inversion contexts, and then to only work on the left-most proposition in the context. This is the way most focused logics are defined, including those by Andreoli, Chaudhuri, and myself in the next chapter. This style of presenting a focusing logic can be called a *fixed* presentation, as the inversion phase is fixed in a particular, though fundamentally arbitrary, shape.

The completeness of focusing for a fixed presentation of focusing is implied by the completeness of focusing for a confluent presentation of the same logic along with the appropriate confluence property for that logic, whereas the reverse is not true. In this sense, the confluent presentation allows us to prove a stronger theorem than the fixed presentation does, though the fixed presentation will be sufficient for our purposes here and in later chapters. We will not prove confluence in this chapter, though doing so is a straightforward exercise.

2.3.9 Running example

Figure 2.9 gives the result of taking our robot example, Figure 2.2, through the polarization process and then running the result through Theorem 2.6. There is indeed only one proof of this focused proposition up to the reordering of invertible rules, and only one proof period if we always decompose invertible propositions in a left-most (i.e., depth-first) ordering as we do in Figure 2.9.

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{
              \dfrac{
                \dfrac{}{\Gamma; \langle \mathsf{6bucks} \rangle \vdash [\mathsf{6bucks}]}\, id^+
                \quad
                \dfrac{
                  \dfrac{
                    \dfrac{
                      \dfrac{
                        \dfrac{}{\Gamma; \langle \mathsf{battery} \rangle \vdash [\mathsf{battery}]}\, id^+
                        \quad
                        \dfrac{
                          \dfrac{
                            \dfrac{
                              \dfrac{}{\Gamma; \langle \mathsf{robot} \rangle \vdash [\mathsf{robot}]}\, id^+
                            }{\Gamma; \langle \mathsf{robot} \rangle \vdash \mathsf{robot}}\, focus_R
                          }{\Gamma; \mathsf{robot} \vdash \mathsf{robot}}\, \eta^+
                        }{\Gamma; [{\uparrow}\mathsf{robot}] \vdash \mathsf{robot}}\, {\uparrow}_L
                      }{\Gamma; \langle \mathsf{battery} \rangle, [\mathsf{battery} \multimap {\uparrow}\mathsf{robot}] \vdash \mathsf{robot}}\, {\multimap}_L
                    }{\Gamma; \mathsf{battery} \multimap {\uparrow}\mathsf{robot}, \langle \mathsf{battery} \rangle \vdash \mathsf{robot}}\, focus_L
                  }{\Gamma; \mathsf{battery} \multimap {\uparrow}\mathsf{robot}, \mathsf{battery} \vdash \mathsf{robot}}\, \eta^+
                }{\Gamma; \mathsf{battery} \multimap {\uparrow}\mathsf{robot}, [{\uparrow}\mathsf{battery}] \vdash \mathsf{robot}}\, {\uparrow}_L
              }{\Gamma; \langle \mathsf{6bucks} \rangle, \mathsf{battery} \multimap {\uparrow}\mathsf{robot}, [\mathsf{6bucks} \multimap {\uparrow}\mathsf{battery}] \vdash \mathsf{robot}}\, {\multimap}_L
            }{\Gamma; \langle \mathsf{6bucks} \rangle, \mathsf{battery} \multimap {\uparrow}\mathsf{robot} \vdash \mathsf{robot}}\, copy
          }{\Gamma; \langle \mathsf{6bucks} \rangle, \mathsf{battery} \multimap {\uparrow}\mathsf{robot} \vdash {\uparrow}\mathsf{robot}}\, {\uparrow}_R
        }{\Gamma; \langle \mathsf{6bucks} \rangle, {\downarrow}(\mathsf{battery} \multimap {\uparrow}\mathsf{robot}) \vdash {\uparrow}\mathsf{robot}}\, {\downarrow}_L
      }{\Gamma; \mathsf{6bucks}, {\downarrow}(\mathsf{battery} \multimap {\uparrow}\mathsf{robot}) \vdash {\uparrow}\mathsf{robot}}\, \eta^+
    }{\Gamma; \mathsf{6bucks} \otimes {\downarrow}(\mathsf{battery} \multimap {\uparrow}\mathsf{robot}) \vdash {\uparrow}\mathsf{robot}}\, \otimes_L
  }{\cdot; !(\mathsf{6bucks} \multimap {\uparrow}\mathsf{battery}), \mathsf{6bucks} \otimes {\downarrow}(\mathsf{battery} \multimap {\uparrow}\mathsf{robot}) \vdash {\uparrow}\mathsf{robot}}\, !_L
}{
  \dfrac{
    \dfrac{
      \dfrac{\cdot; !(\mathsf{6bucks} \multimap {\uparrow}\mathsf{battery}) \otimes \mathsf{6bucks} \otimes {\downarrow}(\mathsf{battery} \multimap {\uparrow}\mathsf{robot}) \vdash {\uparrow}\mathsf{robot}}
      {\cdot; \cdot \vdash !(\mathsf{6bucks} \multimap {\uparrow}\mathsf{battery}) \otimes \mathsf{6bucks} \otimes {\downarrow}(\mathsf{battery} \multimap {\uparrow}\mathsf{robot}) \multimap {\uparrow}\mathsf{robot}}\, {\multimap}_R
    }{\cdot; \cdot \vdash [{\downarrow}(!(\mathsf{6bucks} \multimap {\uparrow}\mathsf{battery}) \otimes \mathsf{6bucks} \otimes {\downarrow}(\mathsf{battery} \multimap {\uparrow}\mathsf{robot}) \multimap {\uparrow}\mathsf{robot})]}\, {\downarrow}_R
  }{\cdot; \cdot \vdash {\downarrow}(!(\mathsf{6bucks} \multimap {\uparrow}\mathsf{battery}) \otimes \mathsf{6bucks} \otimes {\downarrow}(\mathsf{battery} \multimap {\uparrow}\mathsf{robot}) \multimap {\uparrow}\mathsf{robot})}\, focus_R
}\, \otimes_L
$$

Figure 2.9: Proving that a focused transition is possible (where we let $\Gamma = \mathsf{6bucks} \multimap {\uparrow}\mathsf{battery}$)

We have therefore successfully used focusing to get a canonical proof structure that correctly corresponds to our informal series of transitions:

\$6 (1)		*battery (1)*		*robot (1)*
battery-less robot (1)	$\leadsto$	*battery-less robot (1)*	$\leadsto$	*turn \$6 into a battery (all you want)*
turn \$6 into a battery (all you want)		*turn \$6 into a battery (all you want)*		

But at what cost? Figure 2.9 contains a fair amount of bureaucracy compared to the original Figure 2.2, even if does a better job of matching, when read from bottom to top, the series of transitions. A less cluttered way of looking at these proofs is in terms of what we, following Chaudhuri, call *synthetic inference rules* [Cha08].

2.4 Synthetic inference rules

Synthetic inference rules were introduced by Andreoli as the derivation fragments associated with *bipoles*. A monopole is the outermost negative (or positive) structure of a proposition, and a bipole is a monopole surrounded by positive (or, respectively, negative) propositions [And01]. In a polarized setting, bipoles capture the outermost structure of a proposition up to the second occurrence of a shift or an exponential.

The first idea behind synthetic inference rules is that the most important sequents in a polarized sequent calculus are stable sequents where all suspended propositions are atomic. This was reflected by our proof of the completeness of focusing (Theorem 2.6), which was restricted to stable sequents.[7] The second idea is that the bottom-most rule in the proof of a stable sequent must be one of the following:

* $copy$ on some proposition A^- from Γ,
* $focus_L$ on some proposition A^- in Δ, or
* $focus_R$ on the succedent A^+

Once we know which proposition we have focused on, the bipole structure of that proposition (that is, the outermost structure of the proposition up through the second occurrence of a shift of exponential) completely (though not uniquely) dictates the structure of the proof up to the next occurrences of stable sequents.

For example, consider the act of focusing on the proposition $a^+ \multimap {\uparrow}b^+$ in Γ using the $copy$ rule, where a^+ and b^+ are positive atomic propositions. This must mean that a suspended atomic proposition a^+ appears suspended in the context Δ, or else the proof could not be completed:

$$\dfrac{\dfrac{}{\Gamma, a^+ \multimap {\uparrow}b^+; \langle a^+\rangle \vdash [a^+]}\, id^+ \quad \dfrac{\dfrac{\Gamma, a^+ \multimap {\uparrow}b^+; \Delta, \langle b^+\rangle \vdash U}{\Gamma, a^+ \multimap {\uparrow}b^+; \Delta, b^+ \vdash U}\, \eta^+}{\Gamma, a^+ \multimap {\uparrow}b^+; \Delta, [{\uparrow}b^+] \vdash U}\, {\uparrow}_L}{\dfrac{\Gamma, a^+ \multimap {\uparrow}b^+; \Delta, \langle a^+\rangle, [a^+ \multimap {\uparrow}b^+] \vdash U}{\Gamma, a^+ \multimap {\uparrow}b^+; \Delta, \langle a^+\rangle \vdash U}\, copy}\, {\multimap}_L$$

The non-stable sequents in the middle are not interesting parts of the structure of the proof, as they are fully determined by the choice of focus, so we can collapse this series of transitions into a single synthetic rule:

$$\dfrac{\Gamma, a^+ \multimap {\uparrow}b^+; \Delta, \langle b^+\rangle \vdash U}{\Gamma, a^+ \multimap {\uparrow}b^+; \Delta, \langle a^+\rangle \vdash U}\, \mathsf{CP}$$

For the MELL fragment of linear logic, we can associate exactly one rule with every positive proposition (corresponding to a right-focus on that proposition) and two rules with every negative proposition (corresponding to left focus on a negative proposition in the persistent context and left focus on that negative proposition in the positive context). Here are three examples:

$$\dfrac{\Gamma; \Delta, \langle b^+\rangle \vdash U}{\Gamma; \Delta, \langle a^+\rangle, a^+ \multimap {\uparrow}b^+ \vdash U}\, \mathsf{LF}$$

$$\dfrac{\Gamma, A^-; \Delta, \langle b^+\rangle, C^- \vdash D^+}{\Gamma; \Delta \vdash {\downarrow}(!A^- \otimes b^+ \otimes {\downarrow}C^- \multimap {\uparrow}D^+)}\, \mathsf{RF} \qquad \dfrac{}{\Gamma; \langle a^+\rangle \vdash a^+}\, \mathsf{RF}'$$

[7] If we had established the unfocused cut rule discussed in Section 2.3.8 and had then proven the completeness of focusing (Theorem 2.6) for arbitrary inverting sequents, it would have enabled an interpretation that puts all unfocused sequents on similar footing, but that is not our goal here.

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\mathsf{6bucks} \multimap {\uparrow}\mathsf{battery} \;;\; \langle \mathsf{robot} \rangle \vdash \mathsf{robot}}\,\mathsf{RF}'}{\mathsf{6bucks} \multimap {\uparrow}\mathsf{battery} \;;\; \mathsf{battery} \multimap {\uparrow}\mathsf{robot}, \langle \mathsf{battery} \rangle \vdash \mathsf{robot}}\,\mathsf{LF}}{\mathsf{6bucks} \multimap {\uparrow}\mathsf{battery} \;;\; \langle \mathsf{6bucks} \rangle, \mathsf{battery} \multimap {\uparrow}\mathsf{robot} \vdash \mathsf{robot}}\,\mathsf{CP}}{\cdot;\cdot \vdash {\downarrow}(!(\mathsf{6bucks} \multimap {\uparrow}\mathsf{battery}) \otimes \mathsf{6bucks} \otimes {\downarrow}(\mathsf{battery} \multimap {\uparrow}\mathsf{robot}) \multimap {\uparrow}\mathsf{robot})}\,\mathsf{RF}
$$

Figure 2.10: Our running example, presented with synthetic rules

This doesn't mean that there are no choices to be made within focused phases, just that, in MELL, those choices are limited to the way the resources – propositions in Δ – are distributed among the branches of the proof. If we also consider additive connectives, we can identify some number of synthetic rules for each right focus, left focus, or copy. This may be zero, as there's no way to successfully right focus on a proposition like $\mathbf{0} \otimes {\downarrow}{\uparrow}A^+$, and therefore zero synthetic inference rules are associated with this proposition. It may be more than one: there are three ways to successfully right focus on the proposition $a^+ \oplus b^+ \oplus c^+$, and so three synthetic inference rules are associated with this proposition:

$$
\dfrac{}{\Gamma; \langle a^+ \rangle \vdash a^+ \oplus b^+ \oplus c^+} \quad \dfrac{}{\Gamma; \langle b^+ \rangle \vdash a^+ \oplus b^+ \oplus c^+} \quad \dfrac{}{\Gamma; \langle c^+ \rangle \vdash a^+ \oplus b^+ \oplus c^+}
$$

Focused proofs of stable sequents are, by definition, in a 1-to-1 correspondence with proofs using synthetic inference rules. If we look at our running example as a derivation using the example synthetic inference rules presented above (as demonstrated in Figure 2.10), we see that the system takes four steps. The middle two steps, furthermore, correspond precisely to the two steps in our informal description of the robot-battery-store system.

2.5 Hacking the focusing system

Despite the novel treatment of suspended propositions in Section 2.3, the presentation of linear logic given there is equivalent to the presentation in Chaudhuri's dissertation [Cha06], in the sense that the logic gives rise to the same synthetic inference rules. It is *not* a faithful intuitionistic analogue to Andreoli's original presentation of focusing [And92], though the presentation in Pfenning's course notes is [Pfe12].[8] Nor does it have the same synthetic inference rules as the focused presentation used in the framework of ordered logical specifications that we presented in [PS09].

In this section, we will discuss four different presentations of focused sequent calculi that are closely connected to the logic we have just presented. Each system differs significantly in its treatment of positive atomic propositions, the exponential $!A$, and the interaction between them.

* Andreoli's original system, which I name the *atom optimization*, complicates the interpretation of atomic propositions as stand-ins for arbitrary propositions.

[8] We will blur the lines, in this section, between Andreoli's original presentation of focused classical linear logic and Pfenning's adaptation to intuitionistic linear logic. In particular, we will mostly use the notation of Pfenning's presentation, but the observations are equally applicable in Andreoli's focused triadic system.

* A further change to the atom optimization, the *exponential optimization*, complicates the relationship between the focused logic and the unfocused logic.
* The *adjoint logic* of Benton and Wadler [BW96] introduces a new syntactic class of persistent propositions, restricting linear propositions to the linear context and persistent propositions to the persistent context.
* The introduction of *permeable atomic propositions*, a notion (which dates at least back to Girard's LU [Gir93]) that some propositions can be treated as *permeable* between the persistent and linear contexts and that permeable atomic propositions can be introduced to stand for this class of permeable propositions.

The reason we survey these different systems is that they all provide a solution to a pervasive problem encountered when using focused sequent calculi as logical frameworks: the need to allow for synthetic inference rules of the form

$$\frac{\Gamma, p; \Delta, r \vdash C}{\Gamma, p; \Delta, q \vdash C}$$

where p is an atomic proposition in the persistent context that is observed (but not consumed), q is an atomic proposition that is consumed in the transition, and r is an atomic proposition that is generated as the result of the transition. In the kinds of specifications we will be dealing with, the ability to form these synthetic inference rules is critical. In some uses, the persistent resource acts as *permission* to consume q and produce r. In other uses, p represents knowledge that we must currently possess in order to enact a transition. As a concrete example, America's 2010 health care reform law introduced a requirement that restaurant menus include calorie information. This means that, in the near future, we can exchange six bucks for a soup and salad at Panera, but only if we know how many calories are in the meal. The six bucks, soup, and salad remain ephemeral resources like q and r, but the calorie count is persistent. A calorie count is scientific knowledge, which is a resource that is not consumed by the transition.

My justification for presenting Chaudhuri's system as the canonical focusing system for linear logic in Section 2.3 is because it most easily facilitates reasoning about the focused sequent calculus *as a logic*. Internal soundness and completeness properties are established by the cut admissibility and identity expansion theorems (Theorems 2.4 and 2.3), and these theorems are conceptually prior to the soundness and completeness of the focused system relative to the unfocused system (Theorems 2.5 and 2.6). The various modifications we discuss in this section complicate the treatment of focused logics as independently justifiable sequent calculi for linear logic. I suggest in Section 2.5.4 that the last option, the incorporation of permeable atomic propositions, is the most pleasing mechanism for incorporating the structure we desire into a focused presentation of linear logic.

All of the options discussed in this section are compatible with a fifth option, discussed in Section 4.7.1, of avoiding positive propositions altogether and instead changing our view of stable sequents. The proposition $\downarrow a^- \multimap \downarrow b^- \multimap c^-$ is associated with this synthetic inference rule:

$$\frac{\Gamma; \Delta \vdash \langle a^- \rangle \quad \Gamma; \Delta' \vdash \langle b^- \rangle}{\Gamma; \Delta, \Delta', \downarrow a^- \multimap \downarrow b^- \multimap c^- \vdash \langle c^- \rangle}$$

If we can prove a general theorem that the sequent $\Gamma; \Delta \vdash \langle a^- \rangle$ can only be proven if $\Delta = a^-$ or if $\Delta = \cdot$ and $a^- \in \Gamma$, then a^- is a *pseudo-positive* atomic proposition. Proving the succedent $\langle a^- \rangle$ where a^- is pseudo-positive is functionally very similar to proving $[a^+]$ in focus for a positive atomic proposition. This gives us license to treat stable sequents that prove a pseudo-positive proposition not as a stable sequent that appears in synthetic inference rules but as an immediate subgoal that gets folded into the synthetic inference rule. If a^- is pseudo-positive, the persistent proposition $\downarrow a^- \multimap \downarrow b^- \multimap c^-$ can be associated with these two synthetic inference rules:

$$\frac{\Gamma; \Delta \vdash \langle b^- \rangle}{\Gamma; \Delta, \downarrow a^- \multimap \downarrow b^- \multimap c^-, a^- \vdash \langle c^- \rangle} \qquad \frac{\Gamma, a^-; \Delta \vdash \langle b^- \rangle}{\Gamma, a^-; \Delta, \downarrow a^- \multimap \downarrow b^- \multimap c^- \vdash \langle c^- \rangle}$$

The machinery of lax logic introduced in Chapter 3 and the fragment of this logic that forms a logical framework in Chapter 4 make it feasible, in practice, to observe when negative atomic propositions are pseudo-positive.

2.5.1 Atom optimization

Andreoli's original focused system isn't polarized, so propositions that are syntactically invalid in a polarized presentation, like $!(p^+ \otimes q^+)$ or $!p^+$, are valid in his system (we would have to write $!\uparrow(p^+ \otimes q^+)$ and $!\uparrow p^+$). It's therefore possible, in an unpolarized presentation, to use the *copy* rule to copy a positive proposition out of the context and into left focus, but the focus immediately blurs, as in this (intuitionistic) proof fragment:[9]

$$\dfrac{\dfrac{\dfrac{\vdots}{p^+ \otimes q^+; p^+, q^+ \Vdash q^+ \otimes p^+}}{p^+ \otimes q^+; p^+ \otimes q^+ \Vdash q^+ \otimes p^+}\,\otimes_L}{\dfrac{p^+ \otimes q^+; [p^+ \otimes q^+] \Vdash q^+ \otimes p^+}{p^+ \otimes q^+; \cdot \Vdash q^+ \otimes p^+}\,\mathit{copy}}\,\mathit{blur}_L$$

Note that, in the polarized setting, the effect of the blur_L rule is accomplished by the $\downarrow_L$ rule.

Andreoli's system makes a single restriction to the *copy* rule: it cannot apply to a positive atomic proposition in the persistent context. On its own, this restriction would make the system incomplete with respect to unfocused linear logic – there would be no focused proof of $!p^+ \multimap p^+$ – and so Andreoli-style focusing systems restore completeness by creating a second initial sequent for positive atomic propositions that allows a positive right focus on an atomic proposition to succeed if the atomic proposition appears in the persistent context:

$$\frac{}{\Gamma; p^+ \Vdash [p^+]}\,id_1^+ \qquad \frac{}{\Gamma, p^+; \cdot \Vdash [p^+]}\,id_2^+$$

With the second initial rule, we can once again prove $!p^+ \multimap p^+$, and the system becomes

[9] We will use the sequent form $\Gamma; \Delta \Vdash C$ in this section for focused but unpolarized systems. Again, we frequently reference Pfenning's presentation of focused linear logic [Pfe12] as a faithful intuitionistic analogue of Andreoli's system.

complete with respect to unfocused linear logic again.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p^+; \cdot \Vdash [p^+]}\, id_2^+}{p^+; \cdot \Vdash p^+}\, focus_R}{\cdot; !p^+ \Vdash p^+}\, !_L}{\cdot; \cdot \Vdash !p^+ \multimap p^+}\, \multimap_R$$

This modified treatment of positive atoms will be called the *atom optimization*, as it reduces the number of focusing steps that need to be applied: it takes only one right focus to prove $!p^+ \multimap p^+$ in Andreoli's system, but it would take two focusing steps to prove the same proposition in Chaudhuri's system (or to prove $!{\uparrow}p^+ \multimap {\uparrow}p^+$ in the focusing system we have presented).

There seem to be three ways of adapting the atom optimization to a polarized setting. The first approach is to add an initial sequent that directly mimics the one in Andreoli's system, while adding an additional requirement to the *copy* rule that A^- is not a shifted positive atomic proposition:

$$\dfrac{}{\Gamma; \langle A^+ \rangle \vdash [A^+]}\, id^+ \qquad \dfrac{}{\Gamma, {\uparrow}p^+; \cdot \vdash [p^+]}\, id_2^+ \qquad \dfrac{A \neq {\uparrow}p^+ \quad \Gamma, A^-; \Delta, [A^-] \vdash U}{\Gamma, A^-; \Delta \vdash U}\, copy^*$$

The second approach is to extend suspended propositions to the persistent context, add a corresponding rule for right focus, and modify the left rule for ! to notice the presence of a positive atomic proposition:

$$\dfrac{A^- \neq {\uparrow}p^+ \quad \Gamma, A^-; \Delta \vdash U}{\Gamma; \Delta, !A^- \vdash U}\, !_{L1} \qquad \dfrac{\Gamma, \langle p^+ \rangle; \Delta \vdash U}{\Gamma; \Delta, !{\uparrow}p^+ \vdash U}\, !_{L2}$$

$$\dfrac{}{\Gamma; \langle A^+ \rangle \vdash [A^+]}\, id_1^+ \qquad \dfrac{}{\Gamma, \langle A^+ \rangle; \cdot \vdash [A^+]}\, id_2^+$$

The third approach is to introduce a new connective, $\Uparrow$, that can only be applied to positive atomic propositions, just as ! can only be applied to negative propositions. We can initially view this option as equivalent to the previous one by defining $\Uparrow p^+$ as a notational abbreviation for $!{\uparrow}p^+$ and styling rules according to the second approach above:

$$\dfrac{\Gamma; \cdot \vdash p^+}{\Gamma; \cdot \vdash [\Uparrow p^+]}\, \Uparrow_R \qquad \dfrac{\Gamma, \langle p^+ \rangle; \Delta \vdash U}{\Gamma; \Delta, \Uparrow p^+ \vdash U}\, \Uparrow_L \qquad \dfrac{}{\Gamma; \langle A^+ \rangle \vdash [A^+]}\, id_1^+ \qquad \dfrac{}{\Gamma, \langle A^+ \rangle; \cdot \vdash [A^+]}\, id_2^+$$

All three of these options are similar; we will go with the last, as it allows us to preserve the original meaning of $!{\uparrow}p^+$ if that is our actual intent. Introducing the atom optimization as a new connective also allows us to isolate the effects that this new connective has on cut admissibility, identity expansion, and the correctness of focusing; we will consider each in turn.

$$
\dfrac{\dfrac{\dfrac{}{\langle p^+\rangle;\cdot\vdash[p^+]}\,id_2^+\qquad\dfrac{}{\langle p^+\rangle;\cdot\vdash[p^+]}\,id_2^+}{\langle p^+\rangle;\cdot\vdash[p^+\otimes p^+]}\,\otimes_R}{\dfrac{\langle p^+\rangle;\cdot\vdash p^+\otimes p^+}{\cdot;\dot{\uparrow}p^+\vdash p^+\otimes p^+}\,\dot{\uparrow}_L}\,focus_R
\quad\text{vs.}\quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\uparrow A^+;\langle A^+\rangle\vdash[A^+]}\,id_1^+\quad\dfrac{}{\uparrow A^+;\langle A^+\rangle\vdash[A^+]}\,id_1^+}{\uparrow A^+;\langle A^+\rangle,\langle A^+\rangle\vdash[A^+\otimes A^+]}\,\otimes_R}{\uparrow A^+;\langle A^+\rangle,\langle A^+\rangle\vdash A^+\otimes A^+}\,focus_R}{\uparrow A^+;\langle A^+\rangle,A^+\vdash A^+\otimes A^+}\,\eta^+}{\uparrow A^+;\langle A^+\rangle,[\uparrow A^+]\vdash A^+\otimes A^+}\,\uparrow_L}{\uparrow A^+;\langle A^+\rangle\vdash A^+\otimes A^+}\,copy}{\uparrow A^+;A^+\vdash A^+\otimes A^+}\,\eta^+}{\uparrow A^+;[\uparrow A^+]\vdash A^+\otimes A^+}\,\uparrow_L}{\dfrac{\uparrow A^+;\cdot\vdash A^+\otimes A^+}{\cdot;!\uparrow A^+\vdash A^+\otimes A^+}\,!_L}\,copy
$$

Figure 2.11: Substituting A^+ for p^+ in the presence of the atom optimization

Identity expansion There is one new case of identity expansion, which is unproblematic:

$$
\dfrac{\overset{\mathcal{D}}{\Gamma;\Delta,\langle\dot{\uparrow}p^+\rangle\vdash U}}{\Gamma;\Delta,\dot{\uparrow}p^+\vdash U}\,\eta^+
\quad\Longrightarrow\quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma,\langle p^+\rangle;\cdot\vdash[p^+]}\,id_2^+}{\Gamma,\langle p^+\rangle;\cdot\vdash p^+}\,focus_R}{\Gamma,\langle p^+\rangle;\cdot\vdash[\dot{\uparrow}p^+]}\,\dot{\uparrow}_R\qquad\dfrac{\overset{\mathcal{D}}{\Gamma;\Delta,\langle\dot{\uparrow}p^+\rangle\vdash U}}{\Gamma,\langle p^+\rangle;\Delta,\langle\dot{\uparrow}p^+\rangle\vdash U}\,weaken}{\Gamma,\langle p^+\rangle;\Delta\vdash U}\,subst^+}{\Gamma;\Delta,\dot{\uparrow}p^+\vdash U}\,\dot{\uparrow}_L
$$

Even though the identity expansion theorem is unproblematic, we can illuminate one problem with the atom optimization by considering the substitution of arbitrary propositions for atomic propositions. Previously, when we substituted a positive proposition for an atomic proposition, the proof's structure remained fundamentally unchanged – instances of the η^+ rule on p^+ turned into admissible instances of the general identity expansion rule η^+ on A^+. Now, we have to explain what it even means to substitute A^+ for p^+ in $\dot{\uparrow}p^+$, since $\dot{\uparrow}A^+$ is not a syntactically valid proposition; the only obvious candidate seems to be $!\uparrow A^+$. That substitution may require us to change the structure of proofs in a significant way, as shown in Figure 2.11. Immediately before entering into any focusing phase where the id_2^+ rule is used n times on the hypothesis $\langle p^+\rangle$, we need to left-focus on $\uparrow A^+$ n times with the *copy* rule to get n copies of $\langle A^+\rangle$ into the linear context, each of which can be used to replace one of the id_2^+ instances with an instance of id_1^+.

Cut admissibility While we might be willing to sacrifice the straightforward interpretation of atomic propositions as stand-ins for arbitrary propositions, another instance of the same problematic pattern arises when we try to establish the critical cut admissibility theorem for the logic with $\dot{\uparrow}p^+$. Most of the new cases are unproblematic, but trouble arises in part 1 when we cut a

$$
\dfrac{
\dfrac{
\dfrac{}{\Gamma; \cdot \vdash [q^+]}\;id_2^+
\quad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{\Gamma; \langle p^+ \rangle \vdash [p^+]}\;id_1^+
}{\Gamma; \langle p^+ \rangle \vdash p^+}\;focus_R
}{\Gamma; p^+ \vdash p^+}\;\eta^+
}{\Gamma; [\uparrow p^+] \vdash p^+}\;\uparrow_L
}{\Gamma; q^+ \multimap \uparrow p^+ \vdash p^+}\;\multimap_L
}{\Gamma; \cdot \vdash p^+}\;copy
\qquad
\dfrac{
\dfrac{
\dfrac{}{\Gamma, \langle p^+ \rangle; \cdot \vdash [p^+]}\;id_2^+
\quad
\dfrac{}{\Gamma, \langle p^+ \rangle; \cdot \vdash [p^+]}\;id_2^+
}{\Gamma, \langle p^+ \rangle; \cdot \vdash [p^+ \otimes p^+]}\;\otimes_R
}{\Gamma, \langle p^+ \rangle; \cdot \vdash p^+ \otimes p^+}\;focus_R
}{\Gamma; \cdot \vdash p^+ \otimes p^+}\;cut(6b)
$$

$$\Longrightarrow$$

$$
\dfrac{
\begin{array}{c}\vdots \\ \Gamma; \cdot \vdash p^+\end{array}
\quad
\dfrac{
\dfrac{
\begin{array}{c}\vdots \\ \Gamma; \cdot \vdash p^+\end{array}
\quad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{\Gamma; \langle p^+ \rangle \vdash [p^+]}\;id_1^+
\quad
\dfrac{}{\Gamma; \langle p^+ \rangle \vdash [p^+]}\;id_1^+
}{\Gamma; \langle p^+ \rangle, \langle p^+ \rangle \vdash [p^+ \otimes p^+]}\;\otimes_R
}{\Gamma; \langle p^+ \rangle, \langle p^+ \rangle \vdash p^+ \otimes p^+}\;focus_R
}{\Gamma; \langle p^+ \rangle, p^+ \vdash p^+ \otimes p^+}\;\eta^+
}{\Gamma; \langle p^+ \rangle \vdash p^+ \otimes p^+}\;cut(3)
}{\Gamma; p^+ \vdash p^+ \otimes p^+}\;\eta^+
}{\Gamma; \cdot \vdash p^+ \otimes p^+}\;cut(3)
$$

$$(\text{where } \Gamma = q^+ \multimap \uparrow p^+, \langle q^+ \rangle)$$

Figure 2.12: A problematic cut that arises from the introduction of the $\upharpoonleft p^+$ connective

right-focused proof of $\upharpoonleft p^+$ against a proof that is decomposing $\upharpoonleft p^+$ on the left:

$$
\dfrac{
\dfrac{\Gamma; \cdot \vdash p^+}{\Gamma; \cdot \vdash [\upharpoonleft p^+]}\;\upharpoonleft_R
\quad
\dfrac{\Gamma, \langle p^+ \rangle; \Delta \vdash U}{\Gamma; \Delta, \upharpoonleft p^+ \vdash U}\;\upharpoonleft_L
}{\Gamma; \Delta \vdash U}\;cut(1)
$$

We are left needing to prove that $\Gamma; \cdot \vdash p^+$ and $\Gamma, \langle p^+ \rangle; \Delta \vdash U$ proves $\Gamma; \Delta \vdash U$, which does not fit the structure of any of our existing cut principles. It is similar to the statement of part 5 of Theorem 2.4 (if $\Gamma; \cdot \vdash A^-$ and $\Gamma, A^-; \underline{\Delta} \vdash \underline{U}$, then $\Gamma; \underline{\Delta} \vdash \underline{U}$), but the proof is not so straightforward.

To see why this cut is more complicated to prove than part 5 of Theorem 2.4, consider what it will take to reduce the cut in the top half of Figure 2.12. We cannot immediately call the induction hypothesis on the sub-derivation in the right branch, as there is no way to prove $p^+ \otimes p^+$ in focus when $\langle p^+ \rangle$ does not appear (twice) in the linear context. We need to get two suspended $\langle p^+ \rangle$ antecedents in the linear context; then we can replace all the instances of id_2^+ with instances of id_1^+ that use freshly-minted $\langle p^+ \rangle$ antecedents. This can be achieved with repeated application of part 3 of Theorem 2.4, as shown in the bottom half of Figure 2.12.

The minimal extension to cut admissibility (Theorem 2.4) that justifies the atom optimization appears to be the following, where $\langle p^+ \rangle^n$ denotes n copies of the suspended positive proposition $\langle p^+ \rangle$.

Theorem 2.7 (Extra cases of cut admissibility (Theorem 2.4))**.**

6a. If $\Gamma; \cdot \vdash p^+$ *and* $\Gamma, \langle p^+\rangle; \Delta \vdash [B^+]$*, then there exists* n *such that* $\Gamma; \Delta, \langle p^+\rangle^n \vdash [B^+]$*.*

6b. If $\Gamma; \cdot \vdash p^+$ *and* $\Gamma, \langle p^+\rangle; \Delta \vdash U$*, then* $\Gamma; \Delta \vdash U$*.*

6c. If $\Gamma; \cdot \vdash p^+$ *and* $\Gamma, \langle p^+\rangle; \Delta, [B^-] \vdash U$*, then there exists* n *such that* $\Gamma; \Delta, \langle p^+\rangle^n, [B^-] \vdash U$*.*

Proof. Induction on the second given derivation; whenever $focus_R$, $focus_L$ or $copy$ are the last rule in part $6b$, we need to make n calls to part 3 of the cut admissibility lemma, each one followed by a use of the η^+ rule, where n is determined by the inductive call to part $6a$ (for $focus_R$) or $6c$ (for $focus_L$ and $copy$).

The calls to part 3 are justified by the existing induction metric: the principal cut formula p^+ stays the same and the part number gets smaller. □

Correctness of focusing The obvious way of extending erasure for our extended logic is to let $(\dot{\uparrow} p^+)^\circ = {!}p^+$ and to let $(\Gamma, \langle p^+\rangle)^\circ = (\Gamma)^\circ, p^+$. Under this interpretation, the soundness of $\dot{\uparrow}_L$ and $\dot{\uparrow}_R$ has the same structure as the soundness of $!_L$ and $!_R$, and the soundness of id_2^+ in the focused system is established with $copy$ and id in the unfocused system:

$$\dfrac{\dfrac{}{\Gamma^\circ, p^+; p^+ \longrightarrow p^+}\ id}{\Gamma^\circ, p^+; \cdot \longrightarrow p^+}\ copy$$

The extension to the proof of completeness requires two additional cases to deal with $\dot{\uparrow}$, both of which are derivable…

$$\dfrac{\Gamma; \cdot \vdash p^+}{\Gamma; \cdot \vdash {\downarrow}{\uparrow} \ldots {\downarrow}{\uparrow}\dot{\uparrow} p^+}\ \dot{\uparrow}_{uR} \qquad \dfrac{\Gamma, \langle p^+\rangle; \Delta \vdash U}{\Gamma; \Delta, {\uparrow}{\downarrow} \ldots {\downarrow}{\uparrow}\dot{\uparrow} p^+ \vdash U}\ \dot{\uparrow}_{uL}$$

…as well as a case dealing with the situation where we apply $copy$ to the erasure of a persistent suspended proposition. This case reduces to a case of ordinary focal substitution:

$$\dfrac{\Gamma, \langle p^+\rangle; \Delta, \langle p^+\rangle \vdash U}{\Gamma, \langle p^+\rangle; \Delta \vdash U}\ \langle copy\rangle_u \quad = \quad \dfrac{\dfrac{}{\Gamma, \langle p^+\rangle; \cdot \vdash [p^+]}\ id_2^+ \qquad \Gamma, \langle p^+\rangle; \Delta, \langle p^+\rangle \vdash U}{\Gamma, \langle p^+\rangle; \Delta \vdash U}\ subst^+$$

For such a seemingly simple change, the atom optimization adds a surprising amount of complexity to the cut admissibility theorem for focused linear logic. What's more, the three extra cases of cut that we had to introduce were all for the purpose of handling a single problematic case in the proof of part 1 where both derivations were decomposing the principal cut formula $\dot{\uparrow} p^+$.

2.5.2 Exponential optimization

The choice of adding $\dot{\uparrow} p^+$ as a special new connective instead of defining it as ${!}{\uparrow} p^+$ paves the way for us to modify its meaning further. For instance, there turns out to be no internal reason for the $\dot{\uparrow}_R$ rule to lose focus in its premise, even though it is critical that $!_R$ lose focus on its premise;

if we fail to do so propositions like $!(p^+ \otimes q^+) \multimap !(q^+ \otimes p^+)$ will have no proof. We can revise $\Uparrow_R$ accordingly.

$$\frac{\Gamma; \cdot \vdash [p^+]}{\Gamma; \cdot \vdash [\Uparrow p^+]} \Uparrow_R \quad \frac{\Gamma, \langle p^+ \rangle; \Delta \vdash U}{\Gamma; \Delta, \Uparrow p^+ \vdash U} \Uparrow_L \quad \frac{}{\Gamma; \langle A^+ \rangle \vdash [A^+]} id_1^+ \quad \frac{}{\Gamma, \langle A^+ \rangle; \cdot \vdash [A^+]} id_2^+$$

This further optimization can be called the *exponential optimization*, as it, like the atom optimization, potentially reduces the number of focusing phases in a proof. Identity expansion is trivial to modify, and cut admissibility is significantly simpler.

The problematic case of cut is easy to handle in this modified system: we can conclude by case analysis that the first given derivation must prove p^+ in focus using the id_2^+ rule. This, in turn, means that $\langle p^+ \rangle$ must already appear in Γ, so $\Gamma = \Gamma', \langle p^+ \rangle$, and the cut reduces to an admissible instance of contraction.

$$\frac{\dfrac{\dfrac{}{\Gamma', \langle p^+ \rangle; \cdot \vdash [p^+]} id_2^+}{\Gamma', \langle p^+ \rangle; \cdot \vdash [\Uparrow p^+]} \Uparrow_R \quad \dfrac{\Gamma', \langle p^+ \rangle, \langle p^+ \rangle; \Delta \vdash U}{\Gamma', \langle p^+ \rangle; \Delta, \Uparrow p^+ \vdash U} \Uparrow_L}{\Gamma', \langle p^+ \rangle; \Delta \vdash U} cut(1) \quad \Longrightarrow \quad \frac{\Gamma', \langle p^+ \rangle, \langle p^+ \rangle; \Delta \vdash U}{\Gamma', \langle p^+ \rangle; \Delta \vdash U} contract$$

Thus, we no longer need the complicated extra parts $6a$ - $6c$ of cut admissibility in order to prove cut admissibility for a focused system with the exponential optimization.

Because cut and identity hold, we can think of a focused logic with the exponential optimization as being internally sensible. The problem is that this logic is no longer *externally* sensible relative to normal linear logic, because we cannot erase $\Uparrow p^+$ into regular linear logic in a sensible way. Specifically, if we continue to define $(\Uparrow p^+)^\circ$ as $!p^+$, then $\Uparrow q^+ \multimap !(q^+ \multimap \uparrow p^+) \multimap \uparrow\Uparrow p^+$ has no proof in focused linear logic, whereas its erasure, $!q^+ \multimap !(q^+ \multimap p^+) \multimap !p^+$, does have an unfocused proof. In other words, the completeness of focusing (Theorem 2.6) no longer holds under the exponential optimization!

Our focused logic with the exponential optimization has some resemblance to tensor logic [MT10], as well the polarized logic that Girard presented in a note, "On the sex of angels," which first introduced the $\uparrow A^+$ and $\downarrow A^-$ notation to the discussion of polarity [Gir91]. Both of these presentations incorporate a general focus-preserving $\Uparrow A^+$ connective – a positive formula with a positive subformula – in lieu of the focus-interrupting $!A^-$ connective. Both presentations also have the prominent caveat that ! in the unfocused logic necessarily corresponds to $\Uparrow\downarrow$ in the focused logic: it is *not* possible to derive $\Uparrow(A \otimes B) \vdash \Uparrow(B \otimes A)$ in these systems, and no apology is made for this fact, because $\Uparrow\downarrow\uparrow(A \otimes B) \vdash \Uparrow\downarrow\uparrow(B \otimes A)$ holds as expected. We want avoid this route because it gives the shifts too much power: they influence the *existence* of proofs, not just the structure of proofs.[10] This interpretation of shifts therefore threatens our critical ability to intuitively understand and explain linear logic connectives as resources.

There is an easily identifiable class of sequents that obey *separation*, which is the property that positive atomic propositions can be separated into two classes p_l^+ and p_p^+. The *linear* positive

[10]Both the note of Girard and the paper of Melliès and Tabareau see the shifts as a form of negation; therefore, writing from an intuitionistic perspective, they are unconcerned that A^+ has a different meaning of $\downarrow\uparrow A^+$ in their constructive logic. There are many propositions where $\neg\neg A$ is provable even though A is not! This view of shifts as negations seems rather foreign to the erasure-based understanding of shifts we have been discussing, though Zeilberger has attempted to reconcile these viewpoints [Zei08].

propositions p_l^+ are never suspended in the persistent context and never appear as ${\uparrow}p_l^+$, whereas the *persistent* positive propositions p_p^+ are never suspended in the linear context and always appear as ${\uparrow}p_p^+$ inside of other propositions. For sequents and formulas obeying separation, we can use the obvious erasure operation and obtain a proof of the completeness of focusing; this notion of separation was the basis of our completeness result in [PS09]. However, separation is a meta-logical property, something that we observe about a fragment of the logic and not an inherent property of the logic itself. There are many propositions A^+ and A^- that we cannot prove in focused linear logic with the exponential optimization even though $(A^+)^\circ$ and $(A^-)^\circ$ are provable in linear logic, and that makes the exponential optimization unsatisfactory.

The remaining two approaches, adjoint logic and the introduction of permeable atomic propositions, can both be seen as attempts to turn separation into a logical property instead of a meta-logical property.

2.5.3 Adjoint logic

We introduced ${\uparrow}p^+$ as a connective defined as $!{\uparrow}p^+$ – that is, the regular $!A^-$ connective plus a little something extra, the shift. After our experience with modifying the rules of ${\uparrow}$, we can motivate adjoint logic by trying to view ${\uparrow}$ as a more primitive connective – that is, we will try to view ! as ${\uparrow}$ plus a little something extra.

It is frequently observed that the exponential $!A$ of linear logic appears to have two or more parts; the general idea is that ${\uparrow}$ represents just one of those pieces. Accounts of linear logic that follow the judgmental methodology of Martin-Löf [ML96], such as the analysis by Chang et al. [CCP03], emphasize that the regular hypothetical sequent $\Gamma; \Delta \longrightarrow A$ of linear logic is establishing the judgment that A is *true*: we can write $\Gamma; \Delta \longrightarrow A\ \mathit{true}$ to emphasize this. The judgment of *validity*, represented by the judgment $A\ \mathit{valid}$, is defined as truth using no ephemeral resources, and $!A$ is understood as the internalization of judgmental validity:

$$\frac{\Gamma; \cdot \longrightarrow A\ \mathit{true}}{\Gamma \longrightarrow A\ \mathit{valid}}\ \mathit{valid} \qquad \frac{\Delta = \cdot \qquad \Gamma \longrightarrow A\ \mathit{valid}}{\Gamma; \Delta \longrightarrow\ !A\ \mathit{true}}\ !'_R$$

The valid rule is invertible, so if we ever need to prove $\Gamma \longrightarrow A\ \mathit{valid}$, we may asynchronously transition to proving $\Gamma; \cdot \longrightarrow A\ \mathit{true}$. This observation is used to explain why we don't normally consider validity on the right in linear logic. Our more familiar rule for $!_R$ is derivable using these two rules:

$$\frac{\Delta = \cdot \qquad \dfrac{\Gamma; \cdot \vdash A\ \mathit{true}}{\Gamma \vdash A\ \mathit{valid}}\ \mathit{valid}}{\Gamma; \Delta \vdash\ !A\ \mathit{true}}\ !'_R$$

Note that the $!'_R$ rule is not invertible, because it forces the linear context to be empty, which means ! must be positive. The valid rule, on the other hand, is invertible and has an asynchronous or negative character, because it represents the invertible step of deciding to prove that A is *valid* (true without recourse to any ephemeral resources) by proving that it is *true* (in a context with no ephemeral resources). This combination of positive and negative actions explains why $!A^-$ is a positive proposition with a negative subformula, and similarly explains why we must break focus when we reach $!A$ on the right and why we must stop decomposing the proposition when

$$\frac{\Gamma; \cdot \longrightarrow A}{\Gamma \longrightarrow GA}\, G_R \qquad \frac{\Gamma, GA; \Delta, A \longrightarrow C}{\Gamma, GA; \Delta \longrightarrow C}\, G_L \qquad \frac{}{\Gamma, x \longrightarrow x}\, init_x$$

$$\frac{\Gamma, X \longrightarrow Y}{\Gamma \longrightarrow X \supset Y} \supset_R \qquad \frac{\Gamma, X \supset Y \longrightarrow X \quad \Gamma, X \supset Y, Y \longrightarrow Z}{\Gamma, X \supset Y \longrightarrow Z} \supset_L$$

$$\frac{\Gamma, X \supset Y \longrightarrow X \quad \Gamma, X \supset Y, Y; \Delta \longrightarrow C}{\Gamma, X \supset Y; \Delta \longrightarrow C} \supset'_L$$

$$\frac{\Gamma \longrightarrow X}{\Gamma; \cdot \longrightarrow FX}\, F_R \qquad \frac{\Gamma, X; \Delta \longrightarrow C}{\Gamma; \Delta, FX \longrightarrow C}\, F_L \qquad \frac{}{\Gamma; a \longrightarrow a}\, init_a$$

$$\frac{\Gamma; \Delta, A \longrightarrow B}{\Gamma; \Delta \longrightarrow A \multimap B} \multimap_R \qquad \frac{\Gamma; \Delta_A \longrightarrow A \quad \Gamma; \Delta, B \longrightarrow C}{\Gamma; \Delta_A, \Delta, A \multimap B \longrightarrow C} \multimap_L$$

Figure 2.13: Some relevant sequent calculus rules for adjoint logic

we reach $!A$ on the left. The salient feature of the exponential optimization's rules for $\upharpoonleft p^+$, of course, is that they do *not* break focus on the right and that they continue to decompose the proposition on the left (into a suspended proposition $\langle p^+ \rangle$ in the persistent context). This is the reason for arguing that $\upharpoonleft$ captures only the first, purely positive, component of the ! connective.

If the $\upharpoonleft$ connective is the first part of the ! connective, can we characterize the rest of the connective? Giving a reasonable answer necessarily requires a more general account of the $\upharpoonleft$ connective – an *unfocused* logic where it is generally applicable rather than restricted to positive atomic propositions. In other words, to account for the behavior of $\upharpoonleft$, we must give a more primitive logic into which linear logic can be faithfully encoded.

A candidate for a more primitive logic, and one that has tacitly formed the basis of much of my previous work on logic programming and logical specification in substructural logic [PS09, SP11b, SP11a], is *adjoint logic*. Adjoint logic was first characterized by Benton and Wadler as a natural deduction system [BW96] and was substantially generalized by Reed in a sequent calculus setting [Ree09]. The logic generalizes both linear logic and Fairtlough and Mendler's lax logic [FM97] as sub-languages of a common logic, whose propositions come in two syntactically distinct categories that are connected by the adjoint operators F and G:

Persistent propositions	$X, Y, Z ::= GA \mid x \mid X \supset Y \mid X \times Y$
Linear propositions	$A, B, C ::= FX \mid a \mid A \multimap B \mid A \otimes B$

In adjoint logic, persistent propositions X appear in the persistent context Γ and as the succedents of sequents $\Gamma \longrightarrow X$, whereas linear propositions A appear in the linear context Δ and as the succedents of sequents $\Gamma; \Delta \longrightarrow A$. Going back to our previous discussion, this means that persistent propositions are only ever judged to be valid, and that linear propositions are only ever judged to be true. A fragment of the logic is shown in Figure 2.13. Note the similarity between the G_L rule and our unfocused *copy* rule, as well as the similarity between F_R and G_R

in Figure 2.13 and the rules $!_R$ and $valid$ in the previous discussion. Linear logic is recovered as a fragment of adjoint logic by removing all of the persistent propositions except for GA; the usual $!A$ is then definable as FGA.[11]

One drawback of this approach is simply the logistics of giving a fully focused presentation of adjoint logic. We end up with a proliferation of propositions, because the syntactic distinction between X and A is orthogonal to the syntactic distinction between positive and negative propositions. A polarized presentation of adjoint logic would have four syntactic categories: X^+, X^-, A^+, and A^-, with one pair of shifts mediating between X^+ and X^- and another pair of shifts mediating between A^+ and A^-.[12] Given a focused presentation of adjoint logic, however, the separation criteria discussed above can be in terms of the two forms of positive atomic proposition a and x. Positive atomic propositions that are always associated with $\Uparrow$ can be encoded as persistent positive atomic propositions x^+, whereas positive atomic propositions that are never associated with $\Uparrow$ can be encoded as linear positive atomic propositions a^+. The proposition $\Uparrow p^+$ can then be translated as Fx^+, where x^+ is the translation of p^+ as a persistent positive atomic proposition.

Adjoint logic gives one answer to why, in Andreoli-style presentations of linear logic, we can't easily substitute positive propositions for positive atomic propositions when those positive atomic propositions appear suspended in the persistent linear context: because these propositions are actually stand-ins for *persistent* propositions, not for linear propositions, and we are working in a fragment of the logic that has no interesting persistent propositions other than atomic propositions x and the negative inclusion GA back into linear propositions. This effectively captures the structure of the separation requirement (as defined at the end of Section 2.5.2 above) in a logical way, but it makes the structure of persistent atomic propositions rather barren and degenerate, and it places an extra logic, adjoint logic, between the focused system and our original presentation of intuitionistic linear logic.

2.5.4 Permeability

Let us review the problems with our previous attempts to motivate a satisfactory treatment of positive propositions in the persistent context. Andreoli's atom optimization interferes with the structure of cut admissibility. The exponential optimization lacks a good interpretation in unfocused linear logic. The adjoint formulation of linear logic introduces persistent positive propositions as members of a syntactic class X of persistent propositions, a syntactic class that usually lies hidden in between the two right-synchronous and right-asynchronous (that is, positive and negative) halves of the ! connective. This approach works but requires a lot of extra machinery.

Our final attempt to logically motivate a notion of a persistent positive proposition will be based on an analysis of *permeability*. Permeability in classical presentations of linear logic dates back to Girard's LU [Gir93]. In this section, we will motivate permeable atomic propositions in

[11] Lax logic, on the other hand, is recovered by removing all of the linear propositions except for FX; the distinguishing connective of lax logic, $\bigcirc X$, is then definable as GFX.

[12] To make matters worse, in Levy's Call-By-Push-Value language, the programming language formalism that corresponds to polarized logic, $\uparrow$ and $\downarrow$ are characterized as adjoints as well (F and U, respectively), so a fully polarized adjoint logic has *three* distinct pairs of unary connectives that can be characterized as adjoints!

intuitionistic linear logic by first considering a new identity expansion principle that only applies to permeable propositions, a syntactic refinement of the positive propositions.[13]

The admissible identity expansion rules, like the admissible identity rule present in most unfocused sequent calculus systems, help us write down compact proofs. If $F(n) = p_1^+ \otimes \ldots \otimes p_n^+$, then the number of steps in the smallest proof of $\Gamma; F(n) \vdash F(n)$ is in $\Omega(n)$. However, by using the admissible identity expansion rule η^+, we can represent the proof in a compact way:

$$\dfrac{\dfrac{\dfrac{}{\Gamma; \langle F(n) \rangle \vdash [F(n)]}\ id^+}{\Gamma; \langle F(n) \rangle \vdash F(n)}\ focus_R}{\Gamma; F(n) \vdash F(n)}\ \eta^+$$

Permeability as a property of identity expansion

The pattern we want to capture with our new version of identity expansion is the situation where we are trying to prove a sequent like $\Gamma; \Delta \vdash \mathbf{1}$ or $\Gamma; \Delta \vdash {!A^-}$ and we know, by the syntactic structure of Δ, that inversion will empty the linear context. One instance of this pattern is the sequent $\Gamma; G(n) \vdash {!{\uparrow}G(n)}$ where $G(n) = {!p_1^+} \otimes \ldots \otimes {!p_n^+}$. Our goal will be to prove such a sequent succinctly by suspending the proposition $G(n)$ directly in the persistent context just as we did with the proof involving $F(n)$ above. To use these suspended propositions, we introduce a hypothesis rule for positive propositions suspended in the persistent context.

$$\dfrac{}{\Gamma, \langle A^+ \rangle; \cdot \vdash [A^+]}\ id_p^+$$

This rule is, of course, exactly the id_2^+ rule from our discussion of Andreoli's system. There is also a focal substitution principle, Theorem 2.8. This theorem was true in Andreoli's system, but we did not need or discuss it.

Theorem 2.8 (Focal substitution (positive, persistent))**.**
If $\Gamma; \cdot \vdash [A^+]$ *and* $\Gamma, \langle A^+ \rangle; \underline{\Delta'} \vdash \underline{U}$*, then* $\Gamma; \underline{\Delta'} \vdash \underline{U}$.

Proof. Once again, this is a straightforward induction over the second given derivation, as in a proof of regular substitution in a natural deduction system. If the second derivation is the axiom id_p^+ applied to the suspended proposition $\langle A^+ \rangle$ we are substituting for, then the result follows immediately using the first given derivation. □

Given this focal substitution principle, we can consider the class of *permeable* positive propositions. A permeable proposition is one where, when we use the admissible η^+ rule to suspend it in the linear context, we might just as well suspend it in the persistent context, as it decomposes entirely into persistent pieces. In other words, we want a class of propositions A_p^+ such that $\Gamma, \langle A_p^+ \rangle; \Delta \vdash U$ implies $\Gamma; \Delta, A_p^+ \vdash U$; this is the permeable identity expansion property.

[13] Permeable *negative* propositions are relevant to classical linear logic, but the asymmetry of intuitionistic linear logic means that, for now, it is reasonable to consider permeability exclusively as a property of positive propositions. We will consider a certain kind of right-permeable propositions in Chapter 3.

$$\dfrac{\overset{\mathcal{D}}{\Gamma, \langle !A\rangle; \Delta \vdash U}}{\Gamma; \Delta, !A \vdash U}\,\eta_p^+ \quad\Longrightarrow\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, A; [A] \vdash \langle A\rangle}\,\mathit{id}^-}{\Gamma, A; \cdot \vdash \langle A\rangle}\,\mathit{copy}}{\Gamma, A; \cdot \vdash A}\,\eta^-}{\Gamma, A; \cdot \vdash [!A]}\,!_R \qquad \dfrac{\overset{\mathcal{D}}{\Gamma, \langle !A\rangle; \Delta \vdash U}}{\Gamma, A, \langle !A\rangle; \Delta \vdash U}\,\mathit{weaken}}{\Gamma, A; \Delta \vdash U}\,\mathit{subst}_p^+}{\Gamma; \Delta, !A \vdash U}\,!_L$$

$$\dfrac{\overset{\mathcal{D}}{\Gamma, \langle \mathbf{1}\rangle; \Delta \vdash U}}{\Gamma; \Delta, \mathbf{1} \vdash U}\,\eta_p^+ \quad\Longrightarrow\quad \dfrac{\dfrac{\dfrac{}{\Gamma; \cdot \vdash [\mathbf{1}]}\,\mathbf{1}_R \qquad \overset{\mathcal{D}}{\Gamma; \Delta, \langle \mathbf{1}\rangle \vdash U}}{\Gamma; \Delta \vdash U}\,\mathit{subst}_p^+}{\Gamma; \Delta, \mathbf{1} \vdash U}\,\mathbf{1}_L$$

$$\dfrac{\overset{\mathcal{D}}{\Gamma, \langle A \otimes B\rangle; \Delta \vdash U}}{\Gamma; \Delta, A \otimes B \vdash U}\,\eta_p^+ \quad\Longrightarrow\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \langle A\rangle, \langle B\rangle; \cdot \vdash [A]}\,\mathit{id}_p^+ \quad \dfrac{}{\Gamma, \langle A\rangle, \langle B\rangle; \cdot \vdash [B]}\,\mathit{id}_p^+}{\Gamma, \langle A\rangle, \langle B\rangle; \cdot \vdash [A \otimes B]} \qquad \dfrac{\overset{\mathcal{D}}{\Gamma, \langle A\rangle, \langle B\rangle, \langle A \otimes B\rangle; \Delta \vdash U}}{\Gamma, \langle A \otimes B\rangle; \Delta \vdash U}\,\mathit{weaken}}{\Gamma, \langle A\rangle, \langle B\rangle; \Delta \vdash U}\,\mathit{subst}_p^+}{\Gamma, \langle A\rangle; \Delta, B \vdash U}\,\eta_p^+}{\Gamma; \Delta, A, B \vdash U}\,\eta_p^+}{\Gamma; \Delta, A \otimes B \vdash U}\,\otimes_L$$

Figure 2.14: Persistent identity expansion

It is possible to precisely characterize the MELL propositions that are permeable as a syntactic refinement of positive propositions:

$$A_p^+ ::= !A^- \mid \mathbf{1} \mid A_p^+ \otimes B_p^+$$

In full first-order linear logic, $\mathbf{0}$, $A_p^+ \oplus B_p^+$, and $\exists x.A_p^+$ would be included as well; essentially only p^+ and $\downarrow A^-$ are excluded from this fragment.

Theorem 2.9 (Permeable identity expansion)**.** *If* $\Gamma, \langle A_p^+\rangle; \Delta \vdash U$, *then* $\Gamma; \Delta, A_p^+ \vdash U$.

Proof. Induction over the structure of the proposition A_p^+ or A^-. The cases of this proof are represented in Figure 2.14. □

As admissible rules, Theorems 2.8 and 2.9 are written subst_p^+ and η_p^+:

$$\dfrac{\Gamma; \cdot \vdash [A^+] \quad \Gamma, \langle A^+\rangle; \underline{\Delta} \vdash \underline{U}}{\Gamma; \underline{\Delta} \vdash \underline{U}}\,\mathit{subst}_p^+ \qquad \dfrac{\Gamma, \langle A_p^+\rangle; \Delta \vdash U}{\Gamma; \Delta, A_p^+ \vdash U}\,\eta_p^+$$

We can use this persistent identity expansion property to give a compressed proof of our motivating example:

$$\dfrac{\dfrac{\dfrac{\overline{\Gamma; \langle G(n)\rangle \vdash [G(n)]}\ id^+}{\Gamma, \langle G(n)\rangle; \cdot \vdash [!G(n)]}\ !_R}{\Gamma, \langle G(n)\rangle; \cdot \vdash !G(n)}\ focus_R}{\Gamma; G(n) \vdash !G(n)}\ \eta_p^+$$

Permeable atomic propositions

It would have been possible, in the discussion of focused linear logic in Section 2.3, to present identity expansion as conceptually prior to atomic propositions. In such a retelling, the η^+ and η^- rules can be motivated as the necessary base cases of identity expansion when we have propositional variables that stand for unknown positive and negative propositions, respectively. Conversely, we can now present a new class of *permeable* atomic propositions p_p^+ that stand in for arbitrary permeable propositions A_p^+. These add a new base case to permeable identity expansion (Theorem 2.9) that can only be satisfied with an explicit η_p^+ rule:

$$\dfrac{\Gamma, \langle p_p^+\rangle; \Delta \vdash U}{\Gamma; \Delta, p_p^+ \vdash U}\ \eta_p^+$$

Because the permeable propositions are a syntactic refinement of the positive propositions, p_p^+ must be a valid positive atomic proposition as well. This is the revised grammar for intuitionistic MELL with permeable atomic propositions:

$$\begin{aligned} A^+ &::= p^+ \mid p_p^+ \mid {\downarrow}A^- \mid !A^- \mid \mathbf{1} \mid A^+ \otimes B^+ \\ A_p^+ &::= p_p^+ \mid !A^- \mid \mathbf{1} \mid A_p^+ \otimes B_p^+ \\ A^- &::= p^- \mid {\uparrow}A^+ \mid A^+ \multimap B^- \end{aligned}$$

This addition to the logic requires some additions to positive identity expansion, cut admissibility, and completeness, but none of the changes are too severe; we consider each in turn.

Identity expansion The new addition to the language of positive propositions requires us to extend identity expansion with one additional case:

$$\dfrac{\Gamma; \Delta, \langle p_p^+\rangle \vdash U}{\Gamma; \Delta, p_p^+ \vdash U}\ \eta_p^+ \quad \Longrightarrow \quad \dfrac{\dfrac{\overline{\Gamma, \langle p_p^+\rangle; \cdot \vdash [p_p^+]}\ id_p^+ \quad \dfrac{\Gamma; \Delta, \langle p_p^+\rangle \vdash U}{\Gamma, \langle p_p^+\rangle; \Delta, \langle p_p^+\rangle \vdash U}\ weaken}{\Gamma, \langle p_p^+\rangle; \Delta \vdash U}\ subst_p^+}{\Gamma; \Delta, p_p^+ \vdash U}\ \eta_p^+$$

Cut admissibility We must clarify the restriction on cut admissibility for our extended logic. In Theorem 2.4, we required that sequents contain only suspensions of atomic propositions, and in our generalization of cut admissibility, we need to further require that all suspensions in the persistent context Γ be permeable and atomic and that all suspensions in the linear context Δ be non-permeable and atomic. Under this restriction, the proof proceeds much as it did for the system with the exponential optimization.

Correctness of focusing There are two ways we can understand the soundness and completeness of focusing for linear logic extended with permeable atomic propositions. One option is to add a notion of permeable atomic propositions to our core linear logic from Figure 2.1, in which case soundness and completeness are straightforward. Alternatively, we can use our intuition that a permeable proposition A is interprovable with $!A$ and let $(p_p^+)^\circ = !p_p^+$.

The erasure of permeable propositions p_p^+ in the focused logic to $!p_p^+$ in the unfocused logic reveals that permeable propositions, which we motivated entirely from a discussion of identity expansion, are effectively a logical treatment of separation. Rather than $\Uparrow$, a separate proposition that we apply only to positive propositions, permeability is a property intrinsic to a given atomic proposition, much like the proposition's positivity or negativity.

2.6 Revisiting our notation

Andreoli, in his 2001 paper introducing the idea of synthetic inference rules [And01], observed that the atom optimization can lead to an exponential explosion in the number of synthetic rules associated with a proposition. For instance, if $a^+ \otimes b^+ \multimap \uparrow c^+$ appears in Γ, the atom optimization means that the following are all synthetic inference rules for that proposition:

$$\frac{\Gamma; \Delta, \langle c^+ \rangle \vdash U}{\Gamma; \Delta, \langle a^+ \rangle, \langle b^+ \rangle \vdash U} \qquad \frac{\Gamma, \langle a^+ \rangle; \Delta, \langle c^+ \rangle \vdash U}{\Gamma, \langle a^+ \rangle; \Delta, \langle b^+ \rangle \vdash U}$$

$$\frac{\Gamma, \langle b^+ \rangle; \Delta, \langle c^+ \rangle \vdash U}{\Gamma, \langle b^+ \rangle; \Delta, \langle a^+ \rangle \vdash U} \qquad \frac{\Gamma, \langle a^+ \rangle, \langle b^+ \rangle; \Delta, \langle c^+ \rangle \vdash U}{\Gamma, \langle a^+ \rangle, \langle b^+ \rangle; \Delta \vdash U}$$

Andreoli suggests coping with this problem by restricting the form of propositions so that positive atoms never appear in the persistent context. From our perspective, this is a rather unusual recommendation, since it just returns us to linear logic without the atom optimization! The focused system in Section 2.3, which we have argued is a more fundamental presentation (following Chaudhuri), effectively avoid this problem.

However, it's not necessary to view Andreoli's proliferation of rules as a problem with the logic; rather, it is possible to view it merely as a problem of notation. It is already the case that, in writing sequent calculus rules, we tacitly use of a fairly large number of notational conventions, at least relative to Gentzen's original formulation where all contexts were treated as sequences of propositions [Gen35]. For instance, the bottom-up reading of the $\mathbf{1}_R$ rule's conclusion, $\Gamma; \cdot \vdash [\mathbf{1}]$, indicates the presence of an additional premise checking that the linear context is empty, and the conclusion $\Gamma; \Delta_1, \Delta_2 \vdash [A \otimes B]$ of the $\otimes_R$ rule indicates the condition that the context can be split into two parts. In other words, both the conclusion of the $\mathbf{1}_R$ rule and $\otimes_R$ rule, as we normally write them, can be seen as having special *matching constructs* that constrain the shape of the context Δ.[14]

I propose to deal with the apparent proliferation of synthetic rules in a system with the atom optimization by adding a new matching construct for the conclusion of rules. We can say that

[14] More than anything else we have discussed so far, this is a view of inference rules that emphasizes *bottom-up* proof search and proof construction. A view of linear logic that is informed by the inverse method, or top-down proof construction, is bound to look very different (see, for example, [Cha06]).

$$\frac{}{\Gamma; \cdot/p \Longrightarrow p}\ init$$

$$\frac{\Gamma; \cdot \Longrightarrow A}{\Gamma; \cdot \Longrightarrow\ !A}\ !_R \quad \frac{\Gamma, A; \Delta \Longrightarrow C}{\Gamma; \Delta/!A \Longrightarrow C}\ !_L \quad \frac{}{\Gamma; \cdot \Longrightarrow \mathbf{1}}\ \mathbf{1}_R \quad \frac{\Gamma; \Delta \Longrightarrow C}{\Gamma; \Delta/\mathbf{1} \Longrightarrow C}\ \mathbf{1}_L$$

$$\frac{\Gamma; \Delta \Longrightarrow A \quad \Gamma; \Delta \Longrightarrow B}{\Gamma; \Delta_1, \Delta_2 \Longrightarrow A \otimes B}\ \otimes_R \quad \frac{\Gamma; \Delta, A, B \Longrightarrow C}{\Gamma; \Delta/A \otimes B \Longrightarrow C}\ \otimes_L$$

$$\frac{\Gamma; \Delta, A \Longrightarrow B}{\Gamma; \Delta \Longrightarrow A \multimap B}\ \multimap_R \quad \frac{\Gamma; \Delta_1 \Longrightarrow A \quad \Gamma; \Delta_2, B \Longrightarrow C}{\Gamma; \Delta_1, \Delta_2/A \multimap B \Longrightarrow C}\ \multimap_L$$

Figure 2.15: Alternative presentation of intuitionstic linear logic

$\Gamma; \Delta$ matches $\Gamma; \Delta'/\langle p^+\rangle$ either when $\langle p^+\rangle \in \Gamma$ and $\Delta = \Delta'$ or when $\Delta = (\Delta', \langle p^+\rangle)$. We can also iterate this construction, so that $\Gamma; \Delta$ matches $\Gamma; \Delta_n/\langle p_1^+\rangle, \ldots, \langle p_n^+\rangle$ if $\Gamma; \Delta$ matches $\Gamma; \Delta_1/\langle p_1^+\rangle$, $\Gamma; \Delta_1$ matches $\Gamma; \Delta_2/\langle p_2^+\rangle$, ...and $\Gamma; \Delta_{n-1}$ matches $\Gamma; \Delta_n/\langle p_n^+\rangle$. Armed with this notation, we can create a concise synthetic connective that is equivalent to the four of the rules discussed previously:

$$\frac{\Gamma; \Delta, \langle c^+\rangle \vdash U}{\Gamma; \Delta/\langle a^+\rangle, \langle b^+\rangle \vdash U}$$

This modified notation need not be reserved for synthetic connectives; we can also use it to combine the two positive identity rules id_1^+ and id_2^+ (in the exponential-optimized system) or, equivalently, id^+ and id_p^+ (in the system incorporating permeability). Furthermore, by giving $\Gamma; \Delta/A^-$ the obviously analogous meaning, we can fuse the $focus_L$ rule and the $copy$ rule into a single rule that is unconcerned with whether the proposition in question came from the persistent or linear contexts:

$$\frac{}{\Gamma; \cdot/\langle A^+\rangle \vdash [A^+]}\ id^+ \quad \frac{\Gamma; \Delta, [A^-] \vdash U}{\Gamma; \Delta/A^- \vdash U}\ focus_L^*$$

Going yet one more step, we could use this notation to revise the original definition of linear logic in Figure 2.1. The *copy* rule in that presentation sticks out as the only rule that doesn't deal directly with a connective, but we can eliminate it by using the $\Gamma; \Delta/A$ matching construct. The resulting presentation, shown in Figure 2.15, is equivalent to the presentation in Figure 2.1.

Theorem 2.10. $\Gamma; \Delta \longrightarrow C$ *if and only if* $\Gamma; \Delta \Longrightarrow C$.

Proof. The reverse direction is a straightforward induction: each rule in Figure 2.15 can be translated as the related rule in Figure 2.1 along with (potentially) an instance of the $copy$ rule.

The forward direction requires a lemma that the *copy* rule is admissible according to the rules of Figure 2.15; this lemma can be established by straightforward induction. Having established the lemma, the forward direction is a straightforward induction on derivations, applying the admissible rule whenever the *copy* rule is encountered. □

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